UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 12 June, 2018 1:30 pm to 4:30 pm

PAPER 119

CATEGORY THEORY

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State the Yoneda Lemma. Deduce that representable functors are projective in $[\mathcal{C}, \mathbf{Set}]$, and that if $\mathcal{C}$ is small then any $F \colon \mathcal{C} \to \mathbf{Set}$ is an epimorphic image of a (small) coproduct of representables.

A functor $F : \mathcal{C} \to \mathbf{Set}$ is called a *monofunctor* if $F(f)$ is injective for every $f \in \operatorname{mor} \mathcal{C}$. If $\mathcal{C}$ is a small category, show that the following conditions are equivalent:

(a) Every morphism of $\mathcal{C}$ is monic.

(b) Every representable functor $\mathcal{C} \to \mathbf{Set}$ is a monofunctor,

(c) Every functor $\mathcal{C} \to \mathbf{Set}$ is an epimorphic image of a monofunctor.

Under what conditions on $\mathcal{C}$ is every functor $\mathcal{C} \to \mathbf{Set}$ a monofunctor? Justify your answer.

[*You may assume the result that* $\alpha \colon F \to G$ *is an epimorphism in* $[\mathcal{C}, \mathbf{Set}]$ *iff each* $\alpha_A \colon FA \to GA$ *is surjective. For the last part, you may wish to consider the pushout of*

$$\begin{array}{ccc} \mathcal{C}(B,-) & \xrightarrow{\mathcal{C}(f,-)} & \mathcal{C}(A,-) \\ \Big\downarrow{\scriptstyle \mathcal{C}(f,-)} & & \\ \mathcal{C}(A,-) & & \end{array}$$

where $f \colon A \to B$ *is a morphism of* $\mathcal{C}$.]

2 (i) A functor $F \colon \mathcal{C} \to \mathcal{D}$ is called *final* if, for each object B of $\mathcal{D}$, the arrow category $(B \downarrow F)$ is (nonempty and) connected. If F is final, show that for any $G \colon \mathcal{D} \to \mathcal{E}$ each cone under GF has a unique extension to a cone under G, and deduce that if $\mathcal{E}$ has colimits of shape $\mathcal{C}$ then it has colimits of shape $\mathcal{D}$.

(ii) $F \colon \mathcal{C} \to \mathcal{D}$ is called a *discrete fibration* if, given $A \in \operatorname{ob} \mathcal{C}$ and $f \colon B \to FA$ in $\mathcal{D}$, where B may depend on the choice of f, there is a unique $\overline{f} \colon \overline{B} \to A$ in $\mathcal{C}$ with $F\overline{f} = f$. Given a commutative square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \Big\downarrow{\scriptstyle G} & & \Big\downarrow{\scriptstyle H} \\ \mathcal{E} & \xrightarrow{K} & \mathcal{F} \end{array}$$

where G is final and H is a discrete fibration, show that there is a unique $L \colon \mathcal{E} \to \mathcal{D}$ with $HL = K$ and $LG = F$.

3 (i) Explain what is meant by a *reflexive* (parallel) pair of morphisms in a category, and show that coequalizers of reflexive pairs exist in any category with pushouts. Show also that arbitrary finite colimits may be constructed from finite coproducts and coequalizers of reflexive pairs.

(ii) Let $f, g\colon A \rightrightarrows B$ be a reflexive pair in an additive category $\mathcal{C}$. Show that, for any $C \in \text{ob}\,\mathcal{C}$, there is a groupoid $\mathcal{G}$ whose objects (resp. morphisms) are the elements of $\mathcal{C}(C, B)$ (resp. $\mathcal{C}(C, A)$), with domain and codomain maps given by composition with f and g respectively. [*Hint: the composite of* $a\colon u \to v$ *and* $b\colon v \to w$ *in* $\mathcal{G}$ *is* $a - rv + b$, *where* r *is a common splitting for* f *and* g.]

(iii) By considering the submonoid $\{(m, n) \mid m \leqslant n \leqslant 2m\}$ of $\mathbb{N} \times \mathbb{N}$, or otherwise, show that the result in (ii) may fail if we replace 'additive' by 'semi-additive'.

4 (i) Explain what is meant by a *monad* on a category $\mathcal{C}$. Define the Kleisli category $\mathcal{C}_{\mathbb{T}}$ associated with a monad $\mathbb{T}$ on $\mathcal{C}$, and sketch the proof that there is an adjunction $\mathcal{C} \rightleftarrows \mathcal{C}_{\mathbb{T}}$ which is initial in the category of adjunctions inducing $\mathbb{T}$. [*You should verify that* $\mathcal{C}_{\mathbb{T}}$ *is a category and that the mappings you propose are functors, but you need not verify the adjunction in detail.*]

(ii) State a condition on an arbitrary adjunction $\mathcal{C} \rightleftarrows \mathcal{D}$ for the Kleisli comparison functor $\mathcal{C}_{\mathbb{T}} \to \mathcal{D}$ to be part of an equivalence of categories. Deduce that if $\mathbb{T}$ is idempotent, then the comparison from $\mathcal{C}_{\mathbb{T}}$ to the Eilenberg–Moore category $\mathcal{C}^{\mathbb{T}}$ is part of an equivalence.

(iii) Let M be the monoid of order-preserving maps $\mathbb{N} \to \mathbb{N}$, considered as a category with one object. Let $T\colon M \to M$ be the functor defined on morphisms by $Tf(0) = 0$ and $Tf(n) = f(n-1) + 1$ for $n > 0$. Show that T carries a monad structure which is not idempotent, but for which the comparison $M_{\mathbb{T}} \to M^{\mathbb{T}}$ is an isomorphism.

5 (i) Explain what is meant by a *cartesian closed category.* Show that the functor category $[\mathcal{C}, \mathbf{Set}]$ is cartesian closed for any small category $\mathcal{C}$. [*You may use the Special Adjoint Functor Theorem, provided you state it precisely.*]

(ii) An object A of a cartesian closed category is said to be *tiny* if the right adjoint $[A, -]$ of $(-) \times A$ itself has a right adjoint. Show that finite products of tiny objects are tiny.

(iii) If $\mathcal{C}$ has binary coproducts, show that the functor $\mathcal{C}(A, -)$ is tiny in $[\mathcal{C}, \mathbf{Set}]$, for any $A \in \text{ob}\,\mathcal{C}$.

(iv) If $\mathcal{C}$ is Cauchy-complete and has an initial object, show that any tiny object of $[\mathcal{C}, \mathbf{Set}]$ is representable. [*You may assume the result that the irreducible projectives in* $[\mathcal{C}, \mathbf{Set}]$ *— that is, those* F *such that* $[\mathcal{C}, \mathbf{Set}](F, -)$ *preserves coproducts and epimorphisms — are exactly the representable functors.*]

6 (i) Let $\mathcal{C}$ be a semi-additive category with finite products and coproducts. Show that, for any finite family of objects $(A_1, A_2, \ldots, A_n)$, the canonical morphism $c \colon \sum_{i=1}^{n} A_i \to \prod_{i=1}^{n} A_i$ (defined by $\pi_i c \nu_j = \delta_{ij}$) is an isomorphism.

(ii) Now suppose that $\mathcal{C}$ has countable products and coproducts, and that the corresponding morphism $\sum_{i=1}^{\infty} A_i \to \prod_{i=1}^{\infty} A_i$ is always an isomorphism. Show that, for every object A, there exists $z \colon A \to A$ satisfying $z + 1_A = z$. Deduce that if $\mathcal{C}$ is additive it must be degenerate (i.e. equivalent to the trivial category $\mathbf{1}$).

(iii) By considering the category **CSLat** of complete semilattices (that is, posets with arbitrary joins) and join-preserving maps, show that the word 'additive' cannot be weakened to 'semi-additive' in the last sentence of (ii).

END OF PAPER