UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 11 June, 2018 1:30 pm to 4:30 pm

PAPER 118

COMPLEX MANIFOLDS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let M be a complex manifold. Define $\mathcal{A}^{p,q}(M)$, the space of global forms of type (p,q). Define the operators $\partial : \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p+1,q}(M)$ and $\bar{\partial} : \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q+1}(M)$.

(b) Let $\mathcal{P}^n$ be an n-dimensional polydisc. Consider $\alpha \in \mathcal{A}^{p,q}(\mathcal{P}^n)$, $p \geqslant 1$, such that $\partial\alpha = 0$. Show that there exists $\beta \in \mathcal{A}^{p-1,q}(\mathcal{P}^n)$ with $\partial\beta = \alpha$.

[*You may use the $\bar{\partial}$-Poincaré lemma which states that if $\mathcal{P}^n$ is a polydisc then the Dolbeault cohomology $H^{p,q}_{\bar{\partial}}(\mathcal{P}^n) = 0$ for all $q \geqslant 1$.*]

(c) The *Bott-Chern cohomology* group of M is defined to be

$$H^{p,q}_{BC}(M) := \frac{\{\alpha \in \mathcal{A}^{p,q}(M), d\alpha = 0\}}{\partial\bar{\partial}\mathcal{A}^{p-1,q-1}(M)}.$$

Prove that when $M = \mathcal{P}^n$, $H^{p,q}_{BC}(M) = 0$ when $p, q \geqslant 1$.

[*You may use the $\bar{\partial}$-Poincaré lemma as well as the smooth Poincaré lemma, which states that the de-Rham cohomology $H^k_{dR}(\mathcal{P}^n) = 0$ for all $k \geqslant 1$.*]

(d) Let M be a compact Kähler manifold. Construct an isomorphism

$$\phi : H^{p,q}_{BC}(M) \cong H^{p,q}_{\bar{\partial}}(M).$$

2 (a) Let X be a topological space and $\mathcal{F}$ a sheaf of abelian groups. Define the *sheaf cohomology groups* $H^i(X, \mathcal{F})$.

(b) Let X be a smooth manifold and let $\underline{\mathbb{Z}}$ denote the constant sheaf on X wth $\mathbb{Z}$ coefficients. Construct a bijection between the set of isomorphism classes of C^∞ complex line bundles over X and the sheaf cohomology group $H^2(X, \underline{\mathbb{Z}})$.

[*When answering this question and parts (d) and (e) below, you may assume any result stated in the lectures.*]

(c) Fix a point p in a topological space X. Consider the presheaf $\mathbb{C}_p$ which assigns

$$\mathbb{C}_p(U) := \begin{cases} \mathbb{C}, & p \in U \\ 0, & p \notin U \end{cases}$$

with restriction maps $\mathbb{C}_p(U) \to \mathbb{C}_p(V)$ being the identity whenever both groups are $\mathbb{C}$. Verify that $\mathbb{C}_p$ satisfies the sheaf axioms.

(d) Fix an elliptic curve $X = \mathbb{C}/\Lambda$ and let p be a point in X. Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions and consider the canonical morphism of sheaves

$$\mathcal{O}_X \xrightarrow{ev_p} \mathbb{C}_p$$

given by evaluation at the point p. Set $\mathcal{I}_p := \ker(ev_p)$ to be the sheaf kernel of the morphism ev_p. Calculate the sheaf cohomology groups $H^i(X, \mathcal{I}_p)$.

(e) Define the holomorphic line bundles $\mathcal{O}(d)$, $d \in \mathbb{Z}$, over an r-dimensional complex projective space $\mathbb{C}P^r$. Calculate the dimension of $H^0(\mathbb{C}P^1, \mathcal{O}(d))$ as a vector space over $\mathbb{C}$.

3 (a) Let M be a complex manifold and let $\mathcal{L}$ denote a holomorphic line bundle over M. Let h denote a Hermitian metric on $\mathcal{L}$. Define the *fundamental* $(1,1)$ *form* $\omega_{(\mathcal{L},h)}$ associated to $(\mathcal{L}, h)$ [You need not justify that the construction is well-defined]. Define what it means for h to be a *positive Hermitian metric.*

(b) Let M be a compact complex manifold of dimension $n \geqslant 1$. Suppose M admits a holomorphic embedding $i : M \to \mathbb{C}P^r$ into a projective space. Set $\mathcal{L} = i^*(\mathcal{O}(1))$ and let K_M denote the canonical holomorphic line bundle $K_M := \Lambda^n \Omega_M$. Show that $H^n(M, \mathcal{L} \otimes K_M) = 0$.

[*Your solution should not make use of the Kodaira-Nakano vanishing theorem.*]

(c) Let M be a compact Kähler manifold satisfying $H^2(M, \mathcal{O}_M) = 0$. Show that M admits a holomorphic embedding into a projective space.

(d) We say that a three-dimensional compact Kähler manifold M is a *Calabi-Yau threefold* if $K_M \cong \mathcal{O}_M$ and $H^1(M, \mathbb{C}) = 0$. Show that any Calabi-Yau threefold admits a holomorphic embedding into a projective space.

4 (a) Let M be a Kähler manifold and let $\mathcal{A}^k_{M,\mathbb{C}}$ denote the space of globally defined complexified C^∞ differential k-forms. Define the operators $L : \mathcal{A}^k_{M,\mathbb{C}} \to \mathcal{A}^{k+2}_{M,\mathbb{C}}$ and $\Lambda : \mathcal{A}^k_{M,\mathbb{C}} \to \mathcal{A}^{k-2}_{M,\mathbb{C}}$. Suppose that M is also compact. Show that L and Λ induce well-defined operators $L : H^k(M,\mathbb{C}) \to H^{k+2}(M,\mathbb{C})$ and $\Lambda : H^k(M,\mathbb{C}) \to H^{k-2}(M,\mathbb{C})$.

(b) Let M be an n-dimensional compact Kähler manifold. Recall that an element $\alpha \in H^n(M,\mathbb{C})$ is said to be primitive if $L(\alpha) = 0$ in $H^{n+2}(M,\mathbb{C})$. Show that $\alpha \in H^n(M,\mathbb{C})$ is primitive if and only if $\Lambda(\alpha) = 0 \in H^{n-2}(M,\mathbb{C})$.

[*You may assume without proof any theorem or identity stated during lecture.*]

(c) Assume now that M is as in part (b) and that $n = 2$. Let α be a primitive $(1,1)$ form i.e. $L(\alpha) = 0 \in \mathcal{A}^4_{M,\mathbb{C}}$. Show that $*\alpha = -\alpha$.

(d) Let M be as part (c) and denote by Q the Poincaré pairing

$$Q : H^2(M,\mathbb{R}) \otimes_{\mathbb{R}} H^2(M,\mathbb{R}) \to \mathbb{R}$$

Show that if there exists a real two-dimensional subspace $V \subset H^2(M,\mathbb{R})$ on which $Q_{|V} : V \otimes_{\mathbb{R}} V \to \mathbb{R}$ vanishes, then $H^2(M,\mathcal{O}_M) \neq 0$.

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(a) Let M be a compact Kähler manifold and let d denote the de Rham differential on complexified differential forms

$$d : \mathcal{A}^k_{M,\mathbb{C}} \to \mathcal{A}^{k+1}_{M,\mathbb{C}}.$$

Show that any holomorphic p-form $\alpha \in \Gamma(M,\Omega^p_M)$ is d-closed i.e. $d(\alpha) = 0 \in \mathcal{A}^{p+1}_{M,\mathbb{C}}$.

(b) Let M be a complex manifold of complex dimension two and let $\underline{\mathbb{C}}$ denote the constant sheaf with $\mathbb{C}$ coefficients. Prove that there is an exact sequence of sheaves

$$0 \to \underline{\mathbb{C}} \to \mathcal{O}_M \xrightarrow{\partial} \Omega^1_M \xrightarrow{\partial} \Omega^2_M \to 0.$$

(c) Let M be a compact complex manifold of complex dimension two. Show that any holomorphic form $\alpha \in \Gamma(M,\Omega^p_M)$ is d-closed.

(d) Let M be as in part (c). Construct an exact sequence:

$$0 \to H^0(M,\Omega^1_M) \to H^1(M,\mathbb{C}) \to H^1(M,\mathcal{O}_M).$$

END OF PAPER