UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2018 1:30 pm to 4:30 pm

PAPER 115

DIFFERENTIAL GEOMETRY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let M be a smooth manifold. Define the *tangent bundle* of M, and the *Lie bracket* $[X,Y]$ of vector fields $X,Y \in \Gamma(TM)$ on M. Prove that $[X,Y]=0$ if and only if the flows defined by X and Y commute.

Let M^{n+1} be a compact $(n+1)$-dimensional manifold with boundary ∂M. Fix a nowhere-zero volume form $\omega \in \Omega^{n+1}(M)$. Assume that the natural map $H^1_{dR}(M) \to H^1_{dR}(\partial M)$ is injective. Suppose that $X_1,\ldots,X_n \in \Gamma(TM)$ are vector fields on M which

1. are everywhere tangent to the boundary along $TM|_{\partial M}$;
2. are pointwise linearly independent;
3. satisfy $[X_i,X_j]=0$ for each i,j;
4. satisfy $\mathcal{L}_{X_i}(\omega)=0$.

Prove that the 1-form $\eta = \iota_{X_1}\ldots\iota_{X_n}(\omega) \in \Omega^1(M)$ is exact, so $\eta = df$ for some smooth $f : M \to \mathbb{R}$. By considering critical points of f, or otherwise, prove that ∂M cannot be connected. [You may use without proof the relation $[\mathcal{L}_X,\iota_Y](\alpha) = \iota_{[X,Y]}(\alpha)$ for vector fields X and Y and differential forms α.]

Deduce that the vector fields $\partial/\partial\theta_1$ and $\partial/\partial\theta_2$ on the two-dimensional torus $S^1 \times S^1$ do not extend to the solid torus $S^1 \times D^2$ as pointwise-independent commuting volume-preserving vector fields, for any choice of volume form.

2

Let M be a smooth manifold, and $\mathcal{D} \subset TM$ a smooth subbundle of the tangent bundle of M. Define what it means for $\mathcal{D}$ to be *involutive* and what it means for $\mathcal{D}$ to be *integrable.*

State and prove the Frobenius integrability theorem.

Let M be the three-dimensional Lie group of upper triangular matrices

$$M = \left\{ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \quad : \quad x_i \in \mathbb{R} \text{ for } i = 1,2,3 \right\}.$$

We identify $M \cong \mathbb{R}^3$ with co-ordinates (x_1, x_2, x_3) in the obvious way, and hence identify $T_{\mathrm{id}}M \cong \mathbb{R}^3$ where $\mathrm{id} = (0,0,0)$ is the identity element of M.

(i) Compute the left-invariant vector fields E_i associated to the standard basis $\{e_1, e_2, e_3\}$ of $T_{\mathrm{id}}M$, and prove that the distribution $\mathcal{D} = \langle E_1, E_2 \rangle \subset TM$ is not involutive.

(ii) Let $\gamma : [0,1] \to M$ be a smooth curve with $\gamma'(t) = \alpha(t)E_1 + \beta(t)E_2 \in \mathcal{D}_{\gamma(t)}$ for every t. Find an expression for $\gamma(t) = (x(t), y(t), z(t))$ in terms of integrals of the functions α and β.

(iii) Let $p = (P_x, P_y, P_z) \in M$. By writing $\alpha(t) = P_x + af(t)$ and $\beta(t) = P_y + bf(t)$, for appropriate constants a, b and for a suitable smooth function f, or otherwise, deduce there is a curve $\gamma : [0,1] \to M$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = p$ and for which $\gamma'(t) \in \mathcal{D}_{\gamma(t)}$ for every t. [You may assume the existence of a smooth function $f : [0,1] \to \mathbb{R}$ with the property that $\int_0^1 f(t)dt = 0$; $\int_0^1 tf(t)dt = 0$; $\int_0^1 \int_0^s f(t)f(s)dsdt \neq 0$.]

Explain briefly why the property in (iii) demonstrates the non-integrability of $\mathcal{D}$.

3

Let M be a smooth manifold and $\alpha \in \Omega^1(M)$ a differential one-form. For vector fields X, Y on M, prove that

$$d\alpha(X,Y) = X \cdot \alpha(Y) - Y \cdot \alpha(X) - \alpha([X,Y]).$$

Let $E \to M$ be a smooth vector bundle. Define the *curvature* F_A of a connection A on E, and explain why $F_A \in \Omega^2(\mathrm{End}(E))$. Prove that

$$F_A(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \mathrm{End}(E)$$

where you should define the operator $\nabla_\bullet : \Gamma(E) \to \Gamma(E)$.

Define the *induced connection* $A^* \otimes A$ on $\mathrm{End}(E)$. If $\tilde{\nabla}_\bullet$ denotes the corresponding operator on $\Gamma(\mathrm{End}(E))$, and if $\phi \in \mathrm{End}(E)$ and $e \in E$, prove that

$$([\tilde{\nabla}_X, \tilde{\nabla}_Y]\phi)(e) = [\nabla_X, \nabla_Y](\phi(e)) - \phi([\nabla_X, \nabla_Y]e) \ \in E.$$

Deduce that $F_{A^* \otimes A} = 0$ if and only if $F_A = \omega \otimes \mathrm{id}$, for some $\omega \in \Omega^2(M)$. [You may assume that a matrix $M \in \mathrm{Mat}_n(\mathbb{R})$ which commutes with all elements of $\mathrm{Mat}_n(\mathbb{R})$ is a scalar multiple of the identity.]

4

Let (M,g) be a Riemannian manifold. Define the *energy* $E(\gamma)$ of a smooth curve $\gamma : [a,b] \to M$. State and prove a formula for the second variation $E''(0)$ of the energy, where $E(s) = E(\gamma_s)$ and $\gamma_s(t)$ is a family of curves for which $\gamma_0(t)$ is a geodesic.

Suppose now M is closed, oriented and of even dimension, and the sectional curvatures $K(X,Y) = \langle R(X,Y)Y, X\rangle_g$ of M are all strictly positive, for arbitrary non-zero linearly independent vector fields X, Y on M. Let $\gamma = \gamma_0 : S^1 \to M$ be a non-constant closed geodesic on M. Prove that parallel transport around γ fixes a vector v in the hyperplane orthogonal to the tangent vector $\gamma'(p) \in T_{\gamma(p)}M$. By considering an appropriate variation $\{\gamma_s\}_{s \in (-\varepsilon, \varepsilon)}$, show that γ cannot be globally length-minimizing in its smooth isotopy class.

END OF PAPER