UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 1 June, 2018 9:00 am to 12:00 pm

PAPER 114

ALGEBRAIC TOPOLOGY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define what it means for two chain maps to be *chain homotopic*. Show that chain homotopic chain maps induce the same map on homology.

Define $S_*(\Delta^n)$ (the simplicial chain complex of the n-simplex). Check that $d^2 = 0$.

Without assuming any results about cellular homology, show that

$$H_*(S_*(\Delta^n)) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * > 0 \end{cases} \quad .$$

2 Suppose X and Y are topological spaces, and that $f : X \to Y$ is a continuous map. Let $Z = X \times [0,1] \coprod Y/\sim$, where $(x_1, 0) \sim (x_2, 0)$ for all $x_1, x_2 \in X$ and $(x,1) \sim f(x)$ for all $x \in X$.

Show that there is a long exact sequence

$$\cdots \to \widetilde{H}_{*+1}(Z;G) \to H_*(X;G) \xrightarrow{f_*} H_*(Y;G) \to \widetilde{H}_*(Z;G) \to \ldots .$$

where G is any finitely generated abelian group.

Now suppose that X and Y are finite cell complexes. Using the exact sequence above, show that $f_* : H_*(X) \to H_*(Y)$ is an isomorphism if and only if $f_* : H_*(X;\mathbb{Z}/p) \to H_*(Y;\mathbb{Z}/p)$ is an isomorphism for all primes p.

If f is a cellular map, show that Z is a finite cell complex, and express $\widetilde{C}_*^{cell}(Z)$ in terms of $C_*^{cell}(X), C_*^{cell}(Y)$, and the map $f_\# : C_*^{cell}(X) \to C_*^{cell}(Y)$.

3 Suppose W is a compact n-manifold with nonempty boundary, and that $j : W \to S^n$ is a smooth embedding. (In particular, this implies that j is injective, that $j(\partial W)$ has a tubular neighborhood, and that $j(W - \partial W)$ is an open subset of S^n.) Show that $H_*(S^n - j(W))$ does not depend on the choice of the embedding j.

Now suppose M is a closed connected k-manifold and $i : M \to S^n$ is a smooth embedding. Express $H_*(S^n - i(M);\mathbb{Z}/2)$ in terms of $H_*(M;\mathbb{Z}/2)$. [*Hint: use the Gysin sequence.*]

4 If M and N are closed oriented connected n-manifolds, define the *degree* of a map $f : M \to N$. What is the degree of the the antipodal map $A : S^n \to S^n$? Justify your answer.

If $h : S^{2k} \to S^{2k}$ satisfies $h = g \circ f$, where $f : S^{2k} \to \mathbb{RP}^{2k}$ and $g : \mathbb{RP}^{2k} \to S^{2k}$, what are the possible values of $\deg h$?

If $h : S^{2k+1} \to S^{2k+1}$ satisfies $h = g \circ f$, where $f : S^{2k+1} \to \mathbb{RP}^{2k+1}$ and $g : \mathbb{RP}^{2k+1} \to S^{2k+1}$, what are the possible values of $\deg h$?

Suppose that N is a closed oriented connected n-manifold, and that $f : S^n \to N$ has degree p, where p is prime. Show that there is some $k > 0$ such that $p^k x = 0$ for all $x \in H_*(N)$ with $0 < * < n$.

[*Hint: consider homology with coefficients in* $\mathbb{Z}/q$ *for primes* $q \neq p$. *You may assume that a closed manifold is homotopy equivalent to a finite cell complex, so the universal coefficient theorem applies.*]

5 Let $\pi : E \to B$ be an n-dimensional real vector bundle. What is meant by a $\mathbb{Z}/2$ *Thom class* for E? State the *Thom Isomorphism* theorem with $\mathbb{Z}/2$ coefficients. Use it to derive a $\mathbb{Z}/2$ version of the Gysin sequence.

By considering the tautological bundle on $\mathbb{RP}^n$, compute the ring structure of $H^*(\mathbb{RP}^n; \mathbb{Z}/2)$. [*You may take the groups* $H^*(\mathbb{RP}^n; \mathbb{Z}/2)$ *as given, as long as you state them clearly.*]

What is the ring structure of $H^*(\mathbb{RP}^n; \mathbb{Z})$? [*You may take the groups* $H^*(\mathbb{RP}^n; \mathbb{Z})$ *as given, as long as you state them clearly.*]

END OF PAPER