MATHEMATICAL TRIPOS Part III

Wednesday, 6 June, 2018  9:00 am to 12:00 pm

PAPER 113

ALGEBRAIC GEOMETRY

Attempt no more than THREE questions.

There are FOUR questions in total.

The questions carry equal weight.

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STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

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You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.
Let \((X, \mathcal{O}_X)\) denote a variety over an algebraically closed field \(k\) and suppose \(F\) and \(G\) are \(\mathcal{O}_X\)-modules; describe the construction of the tensor product sheaf \(F \otimes_{\mathcal{O}_X} G\), and show that its stalk at a point \(P \in X\) is isomorphic to the tensor product over \(\mathcal{O}_{X,P}\) of the respective stalks.

If \(\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\) is a morphism of varieties over \(k\) and \(F\) is a \(\mathcal{O}_Y\)-module, describe the construction of the pullback \(\mathcal{O}_X\)-module \(\phi^*F\). Prove that \(\phi^*\) is a right exact functor from \(\mathcal{O}_Y\)-modules to \(\mathcal{O}_X\)-modules, i.e. takes each exact sequence \(F \to G \to H \to 0\) of \(\mathcal{O}_Y\)-modules to a corresponding exact sequence of \(\mathcal{O}_X\)-modules.

For \((X, \mathcal{O}_X)\) a variety over \(k\), define what is meant by an \(\mathcal{O}_X\)-module \(F\) being quasi-coherent. For \(\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\) a morphism of varieties over \(k\), show that quasi-coherent \(\mathcal{O}_Y\)-modules pull back to quasi-coherent \(\mathcal{O}_X\)-modules.

Let \(k\) be an algebraically closed field; state (without proof) the universal property satisfied by the product of two prevarieties \((X, \mathcal{O}_X)\) and \((Y, \mathcal{O}_Y)\) over \(k\), and show that this property determines the product prevariety up to isomorphism (you need not prove existence). Define what is meant by a variety over \(k\) and show that the product of two varieties is a variety.

Given now affine varieties \(X, Y\) over \(k\), show that the product variety is an affine variety with coordinate ring \(k[X \times Y] \cong k[X] \otimes_k k[Y]\).

Suppose \(V\) is an affine variety over \(k\) with coordinate ring \(A = k[V]\), and let \(\Delta \subset V \times V\) denote the diagonal subvariety. Let \(I = I(\Delta) \triangleleft k[V \times V] \cong k[V] \otimes_k k[V]\) denote its ideal of definition, considered as an \(A\)-module via the action on the first factor. Show that there is a well defined \(k\)-derivation \(D : A \to I/I^2\) defined by \(D(a) = [1 \otimes a - a \otimes 1]\) for \(a \in A\). Prove that this induces an isomorphism of \(A\)-modules \(\Omega^1_{A/k} \to I/I^2\), where \(\Omega^1_{A/k}\) is the universal \(k\)-derivation of \(A\). [Hint: Consider the map \(A \otimes_k A \to \Omega^1_{A/k}\) given by \(x \otimes y \mapsto xdy\) and its restriction to \(I\).]
Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be an exact sequence of sheaves of abelian groups on a space $X$. Assuming the result that when $\mathcal{F}_1$ is flabby, the maps on sections $\mathcal{F}_2(U) \to \mathcal{F}_3(U)$ are surjective for all open $U \subset X$, deduce that when both $\mathcal{F}_1$ and $\mathcal{F}_2$ are flabby, so too is $\mathcal{F}_3$. Indicate briefly the construction of sheaf cohomology groups via flabby resolutions, and deduce that any flabby sheaf $\mathcal{F}$ has trivial higher cohomology on $X$.

Let $X$ now denote a smooth irreducible curve over an algebraically closed field (so in particular all the local rings are discrete valuation rings). Describe what is meant by the divisor class group $\text{Cl}(X)$. Prove that $\text{Cl}(X)$ is isomorphic to all of the following groups (where $\mathcal{K}_X^*$ denotes the constant multiplicative sheaf of non-zero rational functions and $\mathcal{O}_X^*$ the multiplicative sheaf of nowhere vanishing regular functions on $X$):

(a) $\text{Pic}(X)$, the group of invertible $\mathcal{O}_X$-modules, modulo isomorphism.

(b) The cokernel of the map $H^0(X, \mathcal{K}_X^*) \to H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$.

(c) $H^1(X, \mathcal{O}_X^*)$.

State (without proof) the Resolution Principle for calculating sheaf cohomology groups. Define the concept of an affine morphism $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ between varieties. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, show that $H^i(Y, \phi_\ast \mathcal{F}) \cong H^i(X, \mathcal{F})$ for all $i \geq 0$. [You may assume here any result on the cohomology of quasi-coherent sheaves on affine varieties, if clearly stated, and any result you may need from Question 3.]

Describe the construction of the sheaves $\mathcal{O}_{\mathbb{P}^n}(m)$ on $\mathbb{P}^n(k)$, where $m$ denotes any integer and $k$ is any algebraically closed field. Identify the group of global sections, and show that its dimension as a vector space over $k$ is $\binom{m+n}{n}$. We denote by $h^i(\mathcal{O}_{\mathbb{P}^n}(m))$, for $i \geq 0$, the dimension of $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$ as a vector space over $k$. Suppose we are given the fact that, for any dimension $r$, if $m$ is sufficiently large then $h^i(\mathcal{O}_{\mathbb{P}^r}(m)) = 0$ for all $i > 0$. Using only general results on cohomology, prove by induction that, for $0 < i \neq n$, we have $h^i(\mathcal{O}_{\mathbb{P}^n}(m)) = 0$ for all $m$, and that $h^n(\mathcal{O}_{\mathbb{P}^n}(m)) = \binom{-m-1}{n}$ for all $m$. Explain briefly why these formulae are consistent with the statement of Serre Duality.