UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 6 June, 2018 9:00 am to 12:00 pm

PAPER 113

ALGEBRAIC GEOMETRY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $(X, \mathcal{O}_X)$ denote a variety over an algebraically closed field k and suppose $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules; describe the construction of the tensor product sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$, and show that its stalk at a point $P \in X$ is isomorphic to the tensor product over $\mathcal{O}_{X,P}$ of the respective stalks.

If $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of varieties over k and $\mathcal{F}$ is a sheaf of $\mathcal{O}_Y$-modules, describe the construction of the pullback $\mathcal{O}_X$-module $\phi^* \mathcal{F}$. Prove that ϕ^* is a right exact functor from $\mathcal{O}_Y$-modules to $\mathcal{O}_X$-modules, i.e. takes each exact sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of $\mathcal{O}_Y$-modules to a corresponding exact sequence of $\mathcal{O}_X$-modules.

For $(X, \mathcal{O}_X)$ a variety over k, define what is meant by an $\mathcal{O}_X$-module $\mathcal{F}$ being *quasi-coherent*. For $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ a morphism of varieties over k, show that quasi-coherent $\mathcal{O}_Y$-modules pull back to quasi-coherent $\mathcal{O}_X$-modules.

Given a k-algebra R, a morphism of k-algebras $\alpha : R \to \mathcal{O}_Y(Y)$ and an R-module M, we define an $\mathcal{O}_Y$-module M_Y on Y to be the sheafification of the presheaf of $\mathcal{O}_Y$-modules given by $U \mapsto M \otimes_R \mathcal{O}_Y(U)$, where the morphism $R \to \mathcal{O}_Y(U)$ is the composite of α with restriction. Show that M_Y is a quasi-coherent $\mathcal{O}_Y$-module. If moreover $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of varieties, show that (in obvious notation) $M_X = \phi^* M_Y$.

[*The construction of the sheafification of a presheaf, and its properties, may be assumed in this question, as may standard results from Commutative Algebra.*]

2

Let k be an algebraically closed field; state (without proof) the universal property satisfied by the product of two prevarieties $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ over k, and show that this property determines the product prevariety up to isomorphism (you need not prove existence). Define what is meant by a variety over k and show that the product of two varieties is a variety.

Given now affine varieties X, Y over k, show that the product variety is an affine variety with coordinate ring $k[X \times Y] \cong k[X] \otimes_k k[Y]$.

Suppose V is an affine variety over k with coordinate ring $A = k[V]$, and let $\Delta \subset V \times V$ denote the diagonal subvariety. Let $I = I(\Delta) \triangleleft k[V \times V] \cong k[V] \otimes_k k[V]$ denote its ideal of definition, considered as an A-module via the action on the first factor. Show that there is a well defined k-derivation $D : A \to I/I^2$ defined by $D(a) = [1 \otimes a - a \otimes 1]$ for $a \in A$. Prove that this induces an isomorphism of A-modules $\Omega^1_{A/k} \to I/I^2$, where $d : A \to \Omega^1_{A/k}$ is the universal k-derivation of A. [*Hint: Consider the map* $A \otimes_k A \to \Omega^1_{A/k}$ *given by* $x \otimes y \mapsto x dy$ *and its restriction to* I.]

3

Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be an exact sequence of sheaves of abelian groups on a space X. Assuming the result that when $\mathcal{F}_1$ is flabby, the maps on sections $\mathcal{F}_2(U) \to \mathcal{F}_3(U)$ are surjective for all open $U \subset X$, deduce that when both $\mathcal{F}_1$ and $\mathcal{F}_2$ are flabby, so too is $\mathcal{F}_3$. Indicate briefly the construction of sheaf cohomology groups via flabby resolutions, and deduce that any flabby sheaf $\mathcal{F}$ has trivial higher cohomology on X.

Let X now denote a smooth irreducible curve over an algebraically closed field (so in particular all the local rings are discrete valuation rings). Describe what is meant by the divisor class group $\mathrm{Cl}(X)$. Prove that $\mathrm{Cl}(X)$ is isomorphic to all of the following groups (where $\mathcal{K}_X^*$ denotes the constant multiplicative sheaf of non-zero rational functions and $\mathcal{O}_X^*$ the multiplicative sheaf of nowhere vanishing regular functions on X):

(a) $\mathrm{Pic}(X)$, the group of invertible $\mathcal{O}_X$-modules, modulo isomorphism.

(b) The cokernel of the map $H^0(X, \mathcal{K}_X^*) \to H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$.

(c) $H^1(X, \mathcal{O}_X^*)$.

4

State (without proof) the *Resolution Principle* for calculating sheaf cohomology groups. Define the concept of an *affine morphism* $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ between varieties. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, show that $H^i(Y, \phi_*\mathcal{F}) \cong H^i(X, \mathcal{F})$ for all $i \geqslant 0$. [*You may assume here any result on the cohomology of quasi-coherent sheaves on affine varieties, if clearly stated, and any result you may need from Question 3.*]

Describe the construction of the sheaves $\mathcal{O}_{\mathbf{P}^n}(m)$ on $\mathbf{P}^n(k)$, where m denotes any integer and k is any algebraically closed field. Identify the group of global sections, and show that its dimension as a vector space over k is $\binom{m+n}{n}$. We denote by $h^i(\mathcal{O}_{\mathbf{P}^n}(m))$, for $i \geqslant 0$, the dimension of $H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m))$ as a vector space over k. Suppose we are given the fact that, for any dimension r, if m is sufficiently large then $h^i(\mathcal{O}_{\mathbf{P}^r}(m)) = 0$ for all $i > 0$. Using only general results on cohomology, prove by induction that, for $0 < i \neq n$, we have $h^i(\mathcal{O}_{\mathbf{P}^n}(m)) = 0$ for all m, and that $h^n(\mathcal{O}_{\mathbf{P}^n}(m)) = \binom{-m-1}{n}$ for all m. Explain briefly why these formulae are consistent with the statement of Serre Duality.

END OF PAPER