UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 1 June, 2018 9:00 am to 11:00 am

PAPER 110

EXTREMAL GRAPH THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State the Erdős-Stone theorem.

Let $t, r \in \mathbb{N}$ be given and let (G_n) be a sequence of graphs, such that G_n has order n and size $(1-1/r+o(1))\binom{n}{2}$. Show that if $K_{r+1}(t)$ is not a subgraph of G_n then G_n contains an r-partite subgraph H of minimum degree $(1-1/r+o(1))n$.

Show that if $\chi(F) = r+1$ and G is extremal for F then G itself has minimum degree $(1-1/r+o(1))n$, where $n = |G|$.

Let $r, s \in \mathbb{N}$ be fixed. Show that, for large n, the unique extremal graph for sK_{r+1} (s disjoint copies of K_{r+1}) is $K_{s-1} + T_r(n-s+1)$ (that is, K_{s-1} with every vertex joined to every vertex of a Turán graph on the remaining vertices).

[Hint. As usual, assign each vertex of G to the class in H in which it has fewest neighbours. Consider the cases (a) some vertex of G has more than $o(n)$ neighbours in its own class, applying induction on s, (b) some class contains s independent edges and (c) each class has a set of $2(s-1)$ vertices meeting all edges in its class.]

2

Let G be a graph of order n and let $k_p(G)$ be the number of copies of K_p in G. Let $c \in \mathbb{R}$ and let $f(G) = k_2(G) - ck_3(G)$. Show that, amongst graphs of order n, the function $f(G)$ takes its maximum on some complete multipartite graph.

Deduce that, for $0 \leqslant \theta \leqslant 1$, if G has $(1-\theta)k_2(T_2(n)) + \theta k_2(T_3(n))$ edges then it contains at least $\theta k_3(T_3(n))$ triangles, where $T_p(n)$ is the p-partite Turán graph of order n.

Suppose now that $n \geqslant 4$ is even, and that G has $n^2/4+1$ edges. Show that G has at least $n/2$ triangles.

[Hint. Consider the case where every edge is in some triangle, and the other case. For the latter, apply induction on n: if uv is in no triangles, how many edges can meet uv?]

3

State and prove Szemerédi's Regularity Lemma. [You may assume that if $U' \subset U$ and $W' \subset W$ satisfy $|U'| \geqslant (1-\delta)|U|$ and $|W'| \geqslant (1-\delta)|W|$ then $|d(U', W') - d(U, W)| \leqslant 2\delta$, and also any quantitative form of the Cauchy-Schwarz inequality that you need.]

Let G be a graph of order n with $51n^2/200$ edges. Explain why there exists some $c > 0$ such that $V(G)$ contains disjoint sets U_1, U_2, U_3 with $|U_i| > cn$, $1 \leqslant i \leqslant 3$, and with each pair (U_i, U_j), $1 \leqslant i < j \leqslant 3$, being 10^{-3}–uniform and having density at least 10^{-3}.

4

Recall from the lectures that $c(t) = \inf\{c : e(G) \geqslant c|G| \implies G \succ K_t\}$.

Show that there exists some constant $\beta > 0$ such that $c(t) \geqslant \beta t\sqrt{\log t}$ for large t.

[Standard probabilistic facts may be assumed if stated clearly.]

Show that $c(t) \leqslant 7t\sqrt{\log t}$ for large t.

[You may assume that, for every integer k, if $e(G) \leqslant 11k|G|$ then $G \succ H$ where $|H| \leqslant 11k + 2$ and $2\delta(H) \geqslant |H| + 4k - 1$.]

Show that, if $n \geqslant 4$ and G has n vertices and $2n - 2$ edges, then $G \succ K_4$.

[Hint. Apply induction on n, considering the case where every edge lies in at least two triangles and the other case.]

Does the same statement hold with $2n - 2$ replaced by $2n - 3$?

END OF PAPER