UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 6 June, 2018 9:00 am to 11:00 am

PAPER 109

COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let X be a set with n elements, and $1 \leqslant k \leqslant n$. Let $\mathcal{A} \subset \mathcal{P}(X)$ be a family of sets containing no $(k+1)$-chain $A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_k$. [Thus, if $k = 1$ then $\mathcal{A}$ is a Sperner system.] For $0 \leqslant i \leqslant n$ let $w(i) > 0$, and define the weight of $\mathcal{A}$ as

$$w(\mathcal{A}) = \sum_{A \in \mathcal{A}} w(|A|).$$

(a) Show that if $k = 1$ then we have

$$\sum_{A \in \mathcal{A}} \binom{n}{|A|}^{-1} \leqslant 1,$$

with equality if and only if $\mathcal{A} = X^{(r)}$ for some r, $0 \leqslant r \leqslant n$.

(b) Show that the weight $w(\mathcal{A})$ of $\mathcal{A}$ is at most the sum of the k largest terms in the sequence $u(0), u(1), \ldots, u(n)$, where $u(i) = \binom{n}{i} w(i)$.

(c) Show that no matter what the weights $w(i)$ are, in (b) we have equality for some family $\mathcal{A}$.

(d) Given n and k, for what integers m are there weights $w(i)$ for which there are exactly m families $\mathcal{A}$ of maximal weight?

2

Let X be a set of n elements and, for $i = 1, \ldots, m$, let A_i and B_i be disjoint subsets of X. Suppose that the family Σ of the pairs (A_i, B_i) *separates* the elements of X, i.e. Σ is such that for all $x, y \in X$, $x \neq y$, there is an index i such that x is in one of A_i and B_i, and y is in the other.

(a) Show that if

$$\sum_{i=1}^{m}(|A_i| + |B_i|) \leqslant \lambda mn,$$

where $\lambda \leqslant 1$, then

$$m \geqslant \lceil (\log_2 n)/\lambda \rceil.$$

(b) Show that for $\lambda = 1$ and $n \geqslant 2$ the lower bound in (i) is best possible.

(c) Let $\lambda = 1/2$. Is the lower bound in (a) best possible for $n = 4$? And for $n = 8$?

3

(i) Let A be a subset of the cube Q_n and let C be the initial segment of length $|A|$ in the binary order on Q_n. Show that

$$|\partial_e(C)| \leqslant |\partial_e(A)|.$$

(ii) Write

$$f_n(a) = \min \left\{ |\{xy \in E(G_n) :\ x \in A, y \notin A\}| :\ A \subset V(G_n), |A| = a \right\}$$

for the edge-isoperimetric function of the $n \times n$ grid G_n with n^2 vertices and $2n(n-1)$ edges. Determine $f_n(a)$ for $(b-1)^2 < a \leqslant b^2 - b$, where $b < n/2$.

4

(i) For $a = (a_1, \ldots, a_n)$, with nonnegative integers a_i, write $D(a)$ for the constant term of the Laurent polynomial

$$F(X;a) = \prod_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} \left(1 - \frac{X_i}{X_j}\right)^{a_i}$$

in $X = (X_1, \ldots, X_n)$. Show that

$$D(a) = \binom{m}{a_1, \ldots, a_n},$$

where $m = \sum_{i=1}^{n} a_i$.

(ii) Let p be a prime, and let $b_1, \ldots, b_p \in \mathbb{Z}_p$ be a sequence of (not necessarily distinct) elements with zero sum: $\sum_{i=1}^{p} b_i = 0$. Show that there are enumerations $a_1, \ldots, a_p$ and $c_1, \ldots, c_p$ of the elements of $\mathbb{Z}_p$ such that $b_1 = c_1 - a_1, \ldots, b_p = c_p - a_p$.

END OF PAPER