UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2018 9:00 am to 12:00 pm

PAPER 108

TOPICS IN ERGODIC THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State the pointwise ergodic theorem, and identify the limit in the theorem when the system is ergodic.

[You may use without proof the L^1 version of the mean ergodic theorem.]

Let $K \geqslant 2$ be an integer. Prove that the measure preserving system $(\mathbb{R}/\mathbb{Z}, \mathcal{B}, m, T_K)$ is ergodic, where m is the Lebesgue measure and $T_K(x) = Kx$.

Define what is a *normal number* and prove that there exists a number that is normal in all integer bases greater than 1.

Fix a number $a \geqslant 1$. Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system, and let $f \in L^1(X, \mu)$. Prove that

$$\lim_{n\to\infty} \frac{f(T^n x)}{n^a} = 0$$

for almost every x.

Is there a value of $a < 1$ for which the above statement holds in general?

[*Hint: You may use without proof the following fact. If $(X, \mathcal{B}, \mu, T)$ is an invertible ergodic measure preserving system with the property that for every $\varepsilon > 0$ there is $A \in \mathcal{B}$ with $0 < \mu(A) < \varepsilon$, then for every $k \in \mathbb{Z}_{>0}$ and $\varepsilon > 0$, there is a set $B \in \mathcal{B}$ such that the sets $B, T(B), \ldots, T^{k-1}(B)$ are pairwise disjoint and $\mu(B) > 1/k - \varepsilon$.*]

2

Define what is a sequence of *full density* and *convergence in density*.

State and prove van der Corput's lemma.

Let $P(x) = ax^2 + bx + c$ be a polynomial with $a, b, c \in \mathbb{R}$ and suppose that a is irrational. Use van der Corput's lemma to show that

$$\frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi i m P(n)) \to 0$$

for all integers $m \neq 0$. Conclude that the fractional part of the sequence $an^2 + bn + c$ is equidistributed in $\mathbb{R}/\mathbb{Z}$.

[*You may NOT use Weyl's theorem on equidistribution of polynomials or Furstenberg's theorem on skew products without proof.*]

3

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and let $\xi \subset \mathcal{B}$ be a partition with $H_\mu(\xi) < \infty$. Prove that the sequence $\frac{1}{n} H_\mu(\xi_0^{n-1})$ is monotone non-increasing.

Give the definitions of $h_\mu(T, \xi)$ and $h_\mu(T)$.

State and prove the Shannon–McMillan–Breiman theorem.

[*You may use without proof the martingale convergence theorem and the maximal inequality for conditional information functions.*]

4

State Rudolph's theorem on measures supported on $\mathbb{R}/\mathbb{Z}$ that are simultaneously $T_2 : x \mapsto 2x$ and $T_3 : x \mapsto 3x$ invariant.

State Host's theorem (including the case of non-ergodic systems) and deduce Rudolph's theorem from it.

Show that there is a number $x \in [0, 1)$ that is normal in base 2 but not in base 3.

[*Hint: Show that there is a T_3-invariant measure supported on the middle third Cantor set that has positive entropy.*]

END OF PAPER