UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 8 June, 2018 9:00 am to 12:00 pm

PAPER 107

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let Ω be an open subset of $\mathbb{R}^n$, and let $u \in C^2(\Omega)$ be a harmonic function on Ω.

(a) Derive the mean value formulae, which say that if $B_\rho(y) \subset\subset \Omega$, then

$$u(y) = \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} u = \frac{1}{n\omega_n \rho^{n-1}} \int_{\partial B_\rho(y)} u,$$

where ω_n is the volume of the unit ball $B_1(0)$ of $\mathbb{R}^n$.

Deduce the strong maximum principle for harmonic functions.

(b) If additionally u is non-negative and if $B_{4\rho}(y) \subset\subset \Omega$, prove the Harnack inequality for u, which says that

$$\sup_{B_\rho(y)} u \leqslant 3^n \inf_{B_\rho(y)} u.$$

(c) If $\Omega = \mathbb{R}^n$ and if u is bounded from above *or* from below, show that u must be constant.

(d) Does the result in (*c*) hold if $\Omega = \mathbb{R}^2 \setminus \{0\}$? What if $\Omega = \mathbb{R}^n \setminus \{0\}$ for $n \neq 2$? Justify your answers.

2

(a) Let $L = a^{ij}D_{ij} + b^iD_i + c$ be an elliptic operator on a bounded domain $\Omega \subset \mathbb{R}^n$, where a^{ij}, b^i, c are given functions on Ω and the summation over repeated indices is assumed. Giving the additional hypotheses needed, state and prove the weak maximum principle for a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $Lu \geqslant 0$ in Ω.

(b) Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $\mathcal{A}_\Omega(u)$ be the area of the graph of a function $u \in C^1(\Omega)$, i.e.

$$\mathcal{A}_\Omega(u) = \int_\Omega \sqrt{1 + |Du|^2}.$$

The Euler–Lagrange equation satisfied by a C^1 critical point u of $\mathcal{A}_\Omega$ has the weak form

$$\int_\Omega \frac{D_iuD_i\varphi}{\sqrt{1+|Du|^2}} = 0 \ \text{ for each } \ \varphi \in C_c^1(\Omega).$$

If $u \in C^2(\Omega)$ and u is a critical point of $\mathcal{A}_\Omega$, explain *briefly* why $u \in C^\infty(\Omega)$.

If $u \in C^2(\overline{\Omega})$ and u is a critical point of $\mathcal{A}_\Omega$, show that

$$\sup_\Omega |Du| = \sup_{\partial\Omega} |Du|.$$

[Hint: *Show first that* $v = |Du|^2$ *satisfies a differential inequality of the form* $a^{ij}D_{ij}v + b^iD_iv \geqslant 0$ *in* Ω.]

(c) Let $n \geqslant 2$, $u \in C^2(\mathbb{R}^n \setminus \{0\})$ and suppose that for each bounded open set $\Omega \subset \mathbb{R}^n \setminus \{0\}$, $u|_\Omega$ is a critical point of $\mathcal{A}_\Omega$. Let $\mathcal{G}_u$ be the the graph of u, i.e. $\mathcal{G}_u = \{(x, u(x)) \,:\, x \in \mathbb{R}^n \setminus \{0\}\} \subset \mathbb{R}^{n+1}$. If $\mathcal{G}_u$ is a cone, i.e. if

$$X \in \mathcal{G}_u,\ \lambda > 0 \implies \lambda X \in \mathcal{G}_u,$$

show that $\mathcal{G}_u$ is an n-dimensional plane in $\mathbb{R}^{n+1}$ passing through the origin.

3 Let $\alpha \in (0,1)$ and let Ω be a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^n$. Let $L = a^{ij}D_{ij} + b^i D_i + c$ with $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{\Omega})$ and $a^{ij}(x)\xi^i\xi^j \geqslant \lambda|\xi|^2$ for some constant $\lambda > 0$ and all $\xi \in \mathbb{R}^n$ and $x \in \Omega$, where summation over repeated indices is assumed. Let $u \in C^{2,\alpha}(\overline{\Omega})$ be a solution to the Dirichlet problem

$$\begin{aligned} Lu &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where $f \in C^{0,\alpha}(\overline{\Omega})$.

(a) Giving any additional hypotheses needed, state without proof the Hopf boundary point lemma concerning functions $v \in C^2(\Omega)$ satisfying $Lv \geqslant 0$ in Ω.

Deduce the strong maximum principle for $v \in C^2(\Omega)$ satisfying $Lv \geqslant 0$ in Ω.

(b) State without proof the global Schauder estimate satisfied by u.

(c) If $f \leqslant 0$ and $u \geqslant 0$ in Ω, show that either $u > 0$ in Ω or $u = 0$ in $\overline{\Omega}$.

(d) Let $\Omega' \subset\subset \Omega$. Show that there is a constant $C > 0$ depending only on $n, \alpha, \Omega, \Omega', L$ such that if $u \geqslant 0$ in Ω then (for f not assumed to have a sign)

$$\sup_{\Omega} u \leqslant C \left(\inf_{\Omega'} u + |f|_{0,\alpha;\Omega} \right).$$

[Hint: *Argue by contradiction.*]

4 Let $B_1(0)$ be the open unit ball in $\mathbb{R}^n$ and let $\alpha \in (0,1)$. Show that for each $\delta \in (0,1)$, there is a constant $C = C(n,\alpha,\delta) \in (0,\infty)$ such that if $u \in C^{2,\alpha}(\overline{B_1(0)})$ is a solution to $\Delta u = f$ in $B_1(0)$ for some $f \in C^{0,\alpha}(\overline{B_1(0)})$, then

$$[D^2u]_{\alpha;\, B_{1/2}(0)} \leqslant \delta[D^2u]_{\alpha;\, B_1(0)} + C\left(|u|_{2;\, B_1(0)} + |f|_{0,\alpha;\, B_1(0)}\right).$$

[*You may use without proof Liouville's Theorem: there does not exist a non-constant harmonic function* w *on* $\mathbb{R}^n$ *such that* $[w]_{\alpha;\,\mathbb{R}^n} < \infty$.]

Explain briefly how to deduce from the above result that there is a constant $C = C(n,\alpha)$ such that, for u and f as above,

$$[D^2u]_{\alpha;B_{1/2}(0)} \leqslant C\left(|u|_{2;\, B_1(0)} + |f|_{0,\alpha;\, B_1(0)}\right). \qquad (\star)$$

[*You are not required to give proofs of any additional results needed.*]

Give an example to show that the estimate $(\star)$ cannot be improved to

$$[D^2u]_{\alpha;B_{1/2}(0)} \leqslant C\left(|u|_{2;\, B_1(0)} + |f|_{0;\, B_1(0)}\right)$$

for some constant $C = C(n,\alpha)$.

5 Let Ω be a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^n$ where $\alpha \in (0,1)$, and for $i,j \in \{1,2,\dots,n\}$, let $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{\Omega})$. Let $L : C^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$ be the operator given by

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu,$$

where summation over repeated indices is assumed. Suppose that L is strictly elliptic in Ω.

(a) State the Fredholm alternative concerning the existence, for any given $f \in C^{0,\alpha}(\overline{\Omega})$, of a solution $u \in C^{2,\alpha}(\overline{\Omega})$ to the Dirichlet problem $Lu = f$ in Ω, $u = 0$ on $\partial\Omega$.

(b) Suppose that there is no non-zero function $u \in C^{2,\alpha}(\overline{\Omega})$ satisfying $Lu = 0$ in Ω, $u = 0$ on $\partial\Omega$. Prove that there is a constant $C_1 = C_1(n, \alpha, L, \Omega)$ such that if $u \in C^{2,\alpha}(\overline{\Omega})$ solves the Dirichlet problem $Lu = f$ in Ω, $u = 0$ on $\partial\Omega$ for some $f \in C^{0,\alpha}(\overline{\Omega})$, then

$$|u|_{0;\Omega} \leqslant C_1 |f|_{0,\alpha;\Omega}.$$

[Hint: *Argue by contradiction.*]

Let $\mathcal{Q} : C^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$ be an operator satisfying

$$|\mathcal{Q}(u)|_{0,\alpha;\Omega} \leqslant C|u|^2_{2,\alpha;\Omega} + \delta$$

and

$$|\mathcal{Q}(u_1) - \mathcal{Q}(u_2)|_{0,\alpha;\Omega} \leqslant \left(C\left(|u_1|_{2,\alpha;\Omega} + |u_2|_{2,\alpha;\Omega}\right) + \delta\right)|u_1 - u_2|_{2,\alpha;\Omega}$$

for some constants $C > 0$, $\delta > 0$ and all $u, u_1, u_2 \in \{v \in C^{2,\alpha}(\overline{\Omega}) : |v|_{2,\alpha;\Omega} \leqslant 1\}$.

(c) Suppose that there is no non-zero function $u \in C^{2,\alpha}(\overline{\Omega})$ satisfying $Lu = 0$ in Ω, $u = 0$ on $\partial\Omega$. Prove that for any given $C > 0$, there are positive constants $\delta = \delta(n, \alpha, L, C, \Omega)$ and $\epsilon_0 = \epsilon_0(n, \alpha, L, C, \Omega)$ such that if $\mathcal{Q}$ satisfies the above conditions then for each $f \in C^{0,\alpha}(\overline{\Omega})$ with $|f|_{0,\alpha;\Omega} \leqslant \epsilon_0$, the Dirichlet problem $Lu = \mathcal{Q}(u) + f$ in Ω, $u = 0$ on $\partial\Omega$ has a solution $u \in C^{2,\alpha}(\overline{\Omega})$.

[Hint: *Set up the question as a fixed point problem for a map* $T : \mathcal{B} \to \mathcal{B}$ *where* $\mathcal{B} = \{v \in C^{2,\alpha}(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega \text{ and } |v|_{2,\alpha;\Omega} \leqslant \epsilon\}$ *for an appropriate choice of* $\epsilon > 0$.]

(d) Deduce that there is a constant $\beta = \beta(n, \alpha, \Omega) \in (0,1)$ such that if $\psi \in C^{2,\alpha}(\overline{\Omega})$ satisfies $|\psi|_{2,\alpha;\Omega} \leqslant \beta$, then the minimal surface equation $\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0$ in Ω has a solution $u \in C^{2,\alpha}(\overline{\Omega})$ with $u = \psi$ on $\partial\Omega$.

[*You may use without proof that the minimal surface equation can be written in the form* $\Delta u = \widetilde{\mathcal{Q}}(u)$ *where* $\widetilde{\mathcal{Q}}$ *satisfies* $|\widetilde{\mathcal{Q}}(u)|_{0,\alpha;\Omega} \leqslant C|u|^2_{2,\alpha;\Omega}$ *and* $|\widetilde{\mathcal{Q}}(u_1) - \widetilde{\mathcal{Q}}(u_2)|_{0,\alpha;\Omega} \leqslant C\left(|u_1|_{2,\alpha;\Omega} + |u_2|_{2,\alpha;\Omega}\right)|u_1 - u_2|_{2,\alpha;\Omega}$ *for some constant* C *and all* $u, u_1, u_2 \in \{v \in C^{2,\alpha}(\overline{\Omega}) : |v|_{2,\alpha;\Omega} \leqslant 2\}$.]

6 Let Ω a bounded $C^{2,\alpha}$ domain of $\mathbb{R}^n$ for some $\alpha \in (0,1)$, and let $\psi \in C^{2,\alpha}(\overline{\Omega})$. Let Q be the differential operator defined by

$$Qu = \Delta u - V(u),$$

where $V : \mathbb{R} \to \mathbb{R}$ is a given smooth non-increasing function. Suppose that there are functions $\varphi^{\pm} \in C^2(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ with $Q\varphi^+ \leqslant 0 \leqslant Q\varphi^-$ in Ω and $\varphi^+ \geqslant \psi \geqslant \varphi^-$ on $\partial\Omega$.

(a) Show that there exists a function $u_1 \in C^{2,\alpha}(\overline{\Omega})$ with $\varphi^- \leqslant u_1 \leqslant \varphi^+$ in $\overline{\Omega}$ such that $\Delta u_1 = V(\varphi^-)$ in Ω and $u_1 = \psi$ on $\partial\Omega$.

(b) Deduce that there exists a sequence of functions $u_k \in C^{2,\alpha}(\overline{\Omega})$ with $\varphi^- \leqslant u_1 \leqslant u_2 \leqslant u_3 \leqslant \ldots \leqslant \varphi^+$ in $\overline{\Omega}$ such that $\Delta u_k = V(u_{k-1})$ in Ω, $u_k = \psi$ on $\partial\Omega$ for each $k = 1, 2, 3, \ldots$, where $u_0 = \varphi^-$.

(c) Show that there is a constant $C > 0$ depending on $\varphi^{\pm}, \psi, V$ such that the functions u_k as in (b) satisfy

$$|u_k|_{2,\alpha;\Omega} \leqslant \frac{1}{2}|u_{k-1}|_{2,\alpha;\Omega} + C$$

for each $k = 2, 3, \ldots$.

[*You may use without proof the interpolation inequality that for each* $\epsilon > 0$, *there is* $\gamma(\epsilon)$ *such that for any* $v \in C^{2,\alpha}(\overline{\Omega})$,

$$|v|_{0,\alpha;\Omega} \leqslant \epsilon|v|_{2,\alpha;\Omega} + \gamma(\epsilon)|v|_{0;\Omega}\,]$$

(d) Deduce that there exists $u \in C^{2,\alpha}(\overline{\Omega})$ with $\varphi^- \leqslant u \leqslant \varphi^+$ in Ω such that $Qu = 0$ in Ω and $u = \psi$ on $\partial\Omega$.

END OF PAPER