UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 1 June, 2018 1:30 pm to 4:30 pm

PAPER 106

FUNCTIONAL ANALYSIS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let X be a Banach space. Prove that $f_n \xrightarrow{w^*} f$ in X^* if and only if $f_n(x) \to f(x)$ for all $x \in X$. Prove that if $f_n \xrightarrow{w^*} f$ in X^* and $x_n \to x$ in X, then $f_n(x_n) \to f(x)$. Show further that on a $\|\cdot\|$-compact subset of X^*, the $\|\cdot\|$-topology and w^*-topology coincide.

State and prove the Banach–Alaoglu theorem.

Let X and Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. Prove that T^* is w^*-to-w^*-continuous. Show that T is compact if and only if T^* is w^*-to-$\|\cdot\|$-continuous on bounded subsets of Y^*. [You may assume without proof that T is compact if and only if T^* is compact.]

2

Throughout this question H is a complex Hilbert space with $H \neq \{0\}$.

(a) Let A be a commutative unital C^*-subalgebra of $\mathcal{B}(H)$. State the spectral theorem for A.

(b) Let $T \in \mathcal{B}(H)$ be a normal operator and $K = \sigma(T)$ be the spectrum of T in $\mathcal{B}(H)$. Referring to part (a) if necessary, prove that there is a resolution P of the identity of H over K (also known as a spectral measure) such that

$$T = \int_K \lambda \, \mathrm{d}P(\lambda) \; .$$

What can you say about $P(U)$ for a non-empty, open subset U of K? Prove your claim. [If your claim is already part of your statement in part (a), then you cannot simply refer to that. Otherwise, part (a) can be used in your proof.].

Prove the following statements.

1. If λ is an isolated point of K, then λ is an eigenvalue of T and $P(\{\lambda\})$ is the orthogonal projection onto the eigenspace $\ker(\lambda I - T)$.
2. If K consists of a single point, then T is a scalar multiple of the identity.
3. If $\dim H > 1$, then T has a non-trivial invariant subspace: there is a closed subspace L of H such that $L \neq \{0\}$, $L \neq H$ and $T(L) \subset L$.

[Properties of the integral $\int_K f \, \mathrm{d}P$, where $f \in L_\infty(K)$, can be assumed without proof.]

3

(a) Let A and B be non-empty, disjoint convex subsets of a real locally convex space X. Assume that A is open. State and prove the Hahn–Banach separation theorem for A and B. [You may assume any version of the Hahn–Banach extension theorem.]

Let (x_n) be a sequence in a real Banach space X and let $\varrho > 0$. Show that there exists $f \in S_{X^*}$ with $f(x_n) \geqslant \varrho$ for all $n \in \mathbb{N}$ if and only if $\left\| \sum_{i=1}^{n} t_i x_i \right\| \geqslant \varrho \sum_{i=1}^{n} t_i$ for all $n \in \mathbb{N}$ and for all non-negative real numbers $t_1, \ldots, t_n$.

(b) Describe, without proof, the dual space of $C(K)$, where K is a compact Hausdorff space. Prove that if $f_n \xrightarrow{w} 0$ in $C(K)$, then $f_n^2 \xrightarrow{w} 0$ in $C(K)$ also.

(c) State the commutative Gelfand–Naimark theorem. Prove that there is a unique (up to homeomorphism) compact Hausdorff space K such that the complex Banach space ℓ_∞ is isometrically isomorphic to $C(K)$. Show that K contains a homeomorphic copy of $\mathbb{N}$ with the discrete topology which is dense in K. Show further that every bounded function $\mathbb{N} \to \mathbb{C}$ has a unique extension to a continuous function $K \to \mathbb{C}$.

4

State and prove Mazur's theorem. Show that a w-compact subset of a normed space is bounded in norm.

Let $\mathcal{F}$ be a σ-field on a set Ω. Let X be a separable Banach space equipped with the Borel σ-field generated by the norm-topology. Let $f\colon \Omega \to X$ be a measurable function. Prove that $g\colon \Omega \to \mathbb{R}$ given by $g(\omega) = \|f(\omega)\|$ is measurable. [Hint: First prove that there is a sequence (φ_n) in X^* such that $\|x\| = \sup_n \varphi_n(x)$ for all $x \in X$.]

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $f\colon \Omega \to X$ be a measurable function such that $\int_\Omega \|f(\omega)\| \,\mathrm{d}\mu(\omega) < \infty$. Show that $\varphi \circ f\colon \Omega \to \mathbb{R}$ is μ-integrable for all $\varphi \in X^*$. Let $T_f\colon X^* \to \mathbb{R}$ be the map given by

$$T_f(\varphi) = \int_\Omega \varphi \circ f \,\mathrm{d}\mu \qquad (\varphi \in X^*).$$

Taking for granted the fact that T_f is w^*-continuous, explain briefly why there is a unique element of X, which we denote by $\int_\Omega f \,\mathrm{d}\mu$, satisfying

$$\varphi\Big(\int_\Omega f \,\mathrm{d}\mu\Big) = \int_\Omega \varphi \circ f \,\mathrm{d}\mu \qquad \text{for all } \varphi \in X^* \ .$$

Let X be a separable Banach space, $K \subset X$ be a w-compact set, and $f\colon K \to X$ be the inclusion map given by $f(x) = x$ for all $x \in K$. We equip K with the weak topology and X with the norm-topology. Prove that f is measurable with respect to the Borel σ-fields of K and X. Show further that $\int_\Omega \|f(\omega)\| \,\mathrm{d}\mu(\omega) < \infty$ for any bounded Borel measure μ. Let $T\colon C(K)^* \to X$ be defined by $T(\mu) = \int_K f \,\mathrm{d}\mu$, where we identify $C(K)^*$ with the space of bounded regular Borel measures on K. Prove that T is w^*-to-w-continuous. Given $x \in K$, what is $T(\mu)$ if μ is the point mass at x? By appealing to suitable theorems, deduce that $\overline{\mathrm{conv}} K$ is w-compact.

5

(a) Let A be a commutative unital Banach algebra. Show that every character φ on A is continuous with $\|\varphi\| = 1$. Prove that $x \in A$ is invertible if and only if $\varphi(x) \neq 0$ for all $\varphi \in \Phi_A$.

Let K be a compact Hausdorff space and let A be the algebra $C(K)$ with the supremum norm $\|\cdot\|$. Prove that Φ_A is homeomorphic to K. Let $\|\cdot\|_1$ be another algebra norm on A (not necessarily complete), and let B be the completion of $(A, \|\cdot\|_1)$. Prove that the restriction map $R\colon \Phi_B \to \Phi_A$ defined by $R(\varphi) = \varphi\restriction_A$ is a homeomorphism between Φ_B and a closed subset L of K, where we have identified K with Φ_A. Let $U = K \setminus L$. Show that for any $x \in U$ there exist functions $f, g \in A$ such that $g(x) = 1$, $f = 1$ on L, and $fg = 0$ on K, and deduce that $U = \emptyset$. [Hint: first show that there is an open subset V of K such that $x \in V \subset \overline{V} \subset U$ and apply Urysohn's lemma. Then show that f is invertible in B.] Using that R is surjective, prove that $\|f\| \leqslant \|f\|_1$ for all $f \in A$.

(b) State the Beurling–Gelfand Spectral Radius Formula. Show that $r(x) = \|x\|$ for a hermitian element x of a C^*-algebra. Let A and B be unital C^*-algebras, and let $\theta\colon A \to B$ be a unital $*$-homomorphism. Prove that $\|\theta(x)\| \leqslant \|x\|$ for all $x \in A$. Now assume in addition that θ is injective. Show that $\|\theta(x)\| = \|x\|$ for all $x \in A$. [Hint: For the last part, first show that without loss of generality we may assume that $A = C(K)$ for some compact Hausdorff space K.]

END OF PAPER