UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 31 May, 2018 1:30pm to 4:30 pm

PAPER 105

ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Denote by $Q_r(x) := \left(x - \frac{r}{2}, x + \frac{r}{2}\right)^n$ the open cube of side length r, centred at x in $\mathbb{R}^n$. **Assume** $n < p < \infty$**.**

a) Show that there exists a constant C_1 depending on n, p such that for any $v \in C^1(\mathbb{R}^n)$, the estimate:

$$|\overline{v} - v(y)| \leqslant C_1 r^{1-\frac{n}{p}} \|Dv\|_{L^p(Q_r(x))},$$

holds for all $y \in Q_r(x)$, where:

$$\overline{v} = \frac{1}{|Q_r(x)|} \int_{Q_r(x)} v(z) dz.$$

Deduce that there exists a constant C_2 depending on n, p such that

$$|v(y) - v(x)| \leqslant C_2 r^{1-\frac{n}{p}} \|Dv\|_{L^p(Q_r(x))},$$

holds for all $y \in Q_r(x)$.

b) Let $u \in W^{1,p}(\mathbb{R}^n)$. Show that there exists $u^* \in C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ such that $u = u^*$ almost everywhere. Give, without proof, a counterexample to show that this is not true if $p < n$.

c) Let $u \in W^{1,p}(\mathbb{R}^n)$. Defining u^* as in part *b*), show that u^* is *classically* differentiable at almost every point $x \in \mathbb{R}^n$, and at each such point the classical derivative of u^* equals the weak derivative of u.

[*Hint: you may wish to consider the function* $v(y) := u^*(y) - u^*(x) - Du(x) \cdot (y - x)$ *for some appropriately chosen* x.]

You may assume standard results concerning approximation of Sobolev functions as well as the Lebesgue differentiation theorem in the following form: given $f \in L^p_{loc.}(\mathbb{R}^n)$, *for almost every* $x \in \mathbb{R}^n$*:*

$$\frac{1}{|Q_r(x)|} \int_{Q_r(x)} |f(y) - f(x)|^p \, dy \to 0, \qquad \text{as } r \to 0.$$

2

a) State and prove the Lax–Milgram theorem for a bilinear form B defined on a real Hilbert space H.

b) Let $U \subset \mathbb{R}^n$ be open and bounded, with C^∞ boundary. Let L be the linear differential operator which acts on sufficiently regular functions $u : U \to \mathbb{R}$ by:

$$Lu := -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u$$

where $a^{ij}, b^i, c \in C^\infty(\overline{U})$ and $a^{ij} = a^{ji}$. Consider the Robin boundary value problem:

$$\begin{cases} (L+\lambda)u = f & \text{in } U, \\ \displaystyle\sum_{i,j=1}^{n} a^{ij}u_{x_i}\nu_j + \beta u = 0 & \text{on } \partial U, \end{cases} \qquad (\star)$$

where ν is the outward unit normal to ∂U, $\beta \in C^\infty(\partial U)$ and $\lambda \in \mathbb{R}$.

i) State what it means for L to be *uniformly elliptic.*

ii) Find a weak formulation for $(\star)$. In particular you should demonstrate that $u \in C^2(\overline{U})$ is a classical solution of $(\star)$ if and only if it satisfies your criterion to be a weak solution.

iii) Show that if $\lambda > 0$ is sufficiently large, $(\star)$ admits a unique weak solution $u \in H^1(U)$ for any $f \in L^2(U)$. If you apply the result from part a), you should demonstrate carefully that the hypotheses are satisfied.

[*You may assume that there exists $K > 0$ such that for any $u \in H^1(U)$ the inequality:*

$$\|Tu\|^2_{L^2(\partial U)} \leqslant K \|u\|_{L^2(U)} \|Du\|_{L^2(U)},$$

holds, where $T : H^1(U) \to L^2(\partial U)$ is the trace operator.]

3

Let $U \subset \mathbb{R}^n$ be an open, bounded set, with C^∞ boundary. Let L be the linear differential operator which acts on sufficiently regular functions $u : U \to \mathbb{R}$ by:

$$Lu := -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u,$$

where $a^{ij} = a^{ji}, b^i, c \in C^\infty(\overline{U})$ for $i, j = 1, \ldots, n$, and $a^{ij} = a^{ji}$ satisfy the uniform ellipticity condition.

a) Define the bilinear form $B : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$ associated to the boundary value problem for L with homogeneous Dirichlet conditions, and state what it means for L to be *formally self-adjoint* and *positive*.

b) Assume L is uniformly elliptic, formally self-adjoint and positive.

i) Show that B is a new inner product on $H_0^1(U)$ which defines an equivalent norm to $\|\cdot\|_{H^1(U)}$.

ii) For $l = 0, 1, 2, \ldots$, define a family of bilinear forms by

$$((u, v))_{2l} := (L^l u, L^l v), \qquad ((u, v))_{2l+1} := B[L^l u, L^l v],$$

with $(\cdot, \cdot)$ the standard inner product on $L^2(U)$. Show that $((\cdot, \cdot))_k$ is an inner product on $H_0^1(U) \cap H^k(U)$ which defines an equivalent norm to $\|\cdot\|_{H^k(U)}$ for any $k = 0, 1, 2, \ldots$.

iii) Show that there exists an orthonormal basis for $L^2(U)$, $(w_m)_{m=1}^\infty$, together with a sequence of positive numbers, $(\lambda_m)_{m=1}^\infty$, such that for any $u \in L^2(U)$ we have:

$$u \in H_0^1(U) \cap H^k(U) \qquad \Longleftrightarrow \qquad \sum_{m=1}^{\infty} (\lambda_m)^k (u, w_m)^2 < \infty.$$

c) Let $U_T = (0, T) \times U$; $\Sigma_0 = \{0\} \times U$; $\partial^* U_T = (0, T) \times \partial U$. Consider the following initial-boundary value problem:

$$\begin{cases} u_t + Lu = 0 & \text{in } U_T \\ u = \psi & \text{on } \Sigma_0 \\ u = 0 & \text{on } \partial^* U_T \end{cases}$$

where L is a uniformly elliptic, formally self-adjoint and positive operator on U whose coefficients are independent of t. Such problems are known as *parabolic*. By explicitly constructing the solution in terms of w_m, λ_m, show that there exists $u \in C^\infty(U_T)$ such that:

- For any $t > 0$, $u(t, \cdot) \in C^\infty(\overline{U})$ and $u(t, \cdot) = 0$ on ∂U;
- $u_t + Lu = 0$ holds everywhere in U_T;
- $\lim_{t \to 0} \|u(t, \cdot) - \psi(\cdot)\|_{L^2(U)} = 0$.

[*You may assume any results you require from the course concerning elliptic operators.*]

4

a) Let $U \subset \mathbb{R}^n$ be open and bounded, with C^∞ boundary. Set $U_T = U \times (0,T)$, $\partial^* U_T = \partial U \times [0,T]$ and let $\Sigma_t = U \times \{t\}$ for $t \in [0,T]$. Suppose

$$Lu := -\sum_{i,j=1}^{n} \left(a^{ij}(x,t)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} + c(x,t)u$$

where $a^{ij}, b^i, c \in C^\infty(\overline{U}_T)$ and $a^{ij} = a^{ji}$ satisfy the uniform ellipticity condition.

i) Write down the definition for a weak solution $u \in H^1(U_T)$ of the hyperbolic initial-boundary value problem:

$$\begin{cases} u_{tt} + Lu = f & \text{in } U_T, \\ u = \psi_0, \quad u_t = \psi_1 & \text{on } \Sigma_0, \\ u = 0 & \text{on } \partial^* U_T. \end{cases}$$

where $f \in L^2(U_T)$, $\psi_0 \in H^1_0(U)$, $\psi_1 \in L^2(U)$. You need not justify your answer.

ii) Show that a weak solution, if it exists, is unique.

iii) Suppose that N is the surface $N = \{t = \tau(x)\}$ for some $\tau \in C^1(U)$. If N is everywhere characteristic, find an equation satisfied by τ.

b) Consider the linear equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial \rho}\left((1-\rho^2)\frac{\partial u}{\partial \rho}\right) + \frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial t}\right) + \rho \frac{\partial^2 u}{\partial \rho \partial t} = 0$$

for $u : \mathbb{R}^2 \to \mathbb{R}$.

i) Show that this equation is hyperbolic everywhere in $\mathbb{R}^2$, and find and sketch the characteristic surfaces. *[Hint: In view of the dimension, these will in fact be curves.]*

ii) Suppose that $u \in C^2(\mathbb{R}^2)$. Show that if $u(\rho, 0) = u_t(\rho, 0) = 0$ for $-1 < \rho < 1$, then $u(\rho, t) = 0$ for $-1 < \rho < 1$, $t \in \mathbb{R}$. Comment on the connection with part *b)i)*.

END OF PAPER