UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Friday, 8 June, 2018 9:00 am to 12:00 pm

## PAPER 103

## REPRESENTATION THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

Let $\mathbb{F}$ be a field. Let $n$ be a positive integer and let $\lambda$ and $\mu$ be partitions of $n$. Let $S^\lambda$ be the $\mathbb{F}S_n$-Specht module, and let $M^\mu$ be the $\mathbb{F}S_n$-permutation module (as defined in the lectures). Let $\phi \in \mathrm{Hom}_{\mathbb{F}S_n}(S^\lambda, M^\mu)$ be a non-zero $\mathbb{F}S_n$-homomorphism between $S^\lambda$ and $M^\mu$.

**(i)** Prove that if there exists (an extension) $\Phi \in \mathrm{Hom}_{\mathbb{F}S_n}(M^\lambda, M^\mu)$ such that $\Phi|_{S^\lambda} = \phi$ then $\lambda$ dominates $\mu$. You can use results from the course, if appropriately stated (without proof).

From now on let $\mathbb{F} = \mathbb{C}$ be the field of complex numbers. Denote by $\pi^\mu$ the complex character afforded by $M^\mu$, and denote by $\chi^\lambda$ the irreducible complex character afforded by $S^\lambda$. Let $n$ and $k$ be integers such that $0 \leqslant k \leqslant n/2$.

**(ii)** Prove that $\left\langle \pi^{(n-k,k)} \downarrow_{S_{n-1}}, \chi^\rho \right\rangle = 0$, for all $\rho$ partitions of $n-1$ such that $\ell(\rho) \geqslant 3$. (Here $\ell(\rho)$ denotes the number of (non-zero) parts of $\rho$).

**(iii)** Prove that

$$M^{(n-k,k)} \quad \text{and} \quad \bigoplus_{j=0}^{k} S^{(n-j,j)}$$

are isomorphic as $\mathbb{C}S_n$-modules.

**2**

Let $n, p$ and $r$ be positive integers and let $\lambda$ be a partition of $n$. Denote by $\mathcal{H}(\lambda)$ the multiset of hook lengths of $\lambda$.

**(i)** Prove that if $pr \in \mathcal{H}(\lambda)$ then $\{p, r\} \subseteq \mathcal{H}(\lambda)$. You can use results from the course, if appropriately stated (without proof).

**(ii)** Suppose now that $pr \leqslant n$ and consider the converse of the statement in (i), namely: if $\{p, r\} \subseteq \mathcal{H}(\lambda)$ then $pr \in \mathcal{H}(\lambda)$.

Prove the above statement, or find a counterexample to it.

**(iii)** Let $\mathcal{Y}(\lambda)$ be the Young diagram of $\lambda$ and let $(i,j) \in \mathcal{Y}(\lambda)$. Denote by $H_{(i,j)}(\lambda)$ the $(i,j)$-hook of $\lambda$ and let $h_{i,j}(\lambda) = |H_{(i,j)}(\lambda)|$. Suppose that the hook length $h_{i,j}(\lambda) = pr$. Show that

$$|\{(x,y) \in H_{(i,j)}(\lambda) \ : \ p \mid h_{x,y}(\lambda)\}| = r.$$

**3**

Let $p$ be an odd prime. Let $n, m$ and $k$ be integers such that $0 \leqslant m < p^k$ and such that $n = ap^k + m$ for some $a \in \{1, 2, \ldots, p-1\}$. Let $\lambda$ be a partition of $n$ and let $\gamma := C_{p^k}(\lambda)$ be the $p^k$-core of $\lambda$. Denote by $\mathcal{H}(\lambda)$ the multiset of hook lengths of $\lambda$. Prove, or find a counterexample to, each of the following statements.

**(i)** $p$ does not divide $\chi^\lambda(1)$ if and only if $w_{p^k}(\lambda) = a$ and $p$ does not divide $\chi^\gamma(1)$.

**(ii)** If $ap^k \in \mathcal{H}(\lambda)$ then $p$ does not divide $\chi^\lambda(1)$.

**(iii)** If $p$ does not divide $\chi^\lambda(1)$ then $ap^k \in \mathcal{H}(\lambda)$.

You may use any result from the lectures but should include a clear statement of such a result.

**4**

Given a natural number $N$, we let $\nu_p(N)$ be such that $N = p^{\nu_p(N)} \cdot M$, for some number $M$ coprime to $p$.

Let $n = \sum_{k \geqslant 0} \alpha_k p^k$ be the $p$-adic expansion of $n$, where $0 \leqslant \alpha_j \leqslant p-1$ for all $j \in \mathbb{N}_0$. Let $\lambda$ be a partition of $n$ and denote by $\mathcal{H}(\lambda)$ the multiset of hook lengths of $\lambda$. Let $T^C(\lambda)$ be the $p$-core tower of $\lambda$ and let $T^Q(\lambda)$ be the $p$-quotient tower of $\lambda$. It is known that:

$$\nu_p(\chi^\lambda(1)) = \big(\sum_{k \geqslant 0} |T^C(\lambda)_k| - \sum_{k \geqslant 0} \alpha_k\big)/(p-1).$$

A key step in the proof of the above identity is to show that:

$$(\star) \qquad \nu_p\big(\prod_{h \in \mathcal{H}(\lambda)} h\big) \;=\; \sum_{r \geqslant 1} |T^Q(\lambda)_r|.$$

**(i)** Prove that the identity in $(\star)$ holds. You can use previous results from the course, if appropriately stated (without proof).

Let $p$ be a prime number. Let $m, w$ and $n$ be positive integers such that $n = wp + m$. Let $\gamma$ be a $p$-core partition of $m$. Let $B$ and $C$ be the sets defined by:

$$B = \{\chi^\lambda \in \mathrm{Irr}(S_{wp}) \;:\; C_p(\lambda) = \emptyset\} \quad \text{and} \quad C = \{\chi^\mu \in \mathrm{Irr}(S_n) \;:\; C_p(\mu) = \gamma\}.$$

**(ii)** Suppose that $0 \leqslant m < p$. Prove that $|B \cap \mathrm{Irr}_{p'}(S_{wp})| = |C \cap \mathrm{Irr}_{p'}(S_n)|$ and that $|B| = |C|$.

**(iii)** Suppose that $m \geqslant p$. Compute $|C \cap \mathrm{Irr}_{p'}(S_n)|$.

## END OF PAPER