UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2018 1:30 pm to 4:30 pm

PAPER 102

LIE ALGEBRAS AND THEIR REPRESENTATIONS

*Attempt **ALL** questions.*

*There are **FIVE** questions in total.*

Questions 1 and 4 are worth 15 points each.
Question 2 is worth 30 points.
Questions 3 and 5 are worth 20 points each.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
Triangular graph paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State 3 equivalent conditions for a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ to be *semisimple*, briefly defining any terms you use.

Now let $\mathfrak{h}$ be a 3-dimensional Lie algebra over $\mathbb{C}$ with basis x, y, z, and bracket determined by $[xy] = z, [xz] = [yz] = 0$. [Note that you do not need to prove $\mathfrak{h}$ forms a Lie algebra.]

(b) Is $\mathfrak{h}$ semisimple? Prove or disprove it. [You may use any of the equivalent conditions from part (a), but prove any other result you use.]

(c) Is every finite-dimensional representation of $\mathfrak{h}$ completely reducible? Prove it or give a counterexample.

2

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ of rank ℓ with Cartan subalgebra $\mathfrak{t}$ and corresponding root system Φ. Let Δ be a root basis in Φ, and let Φ^+ be the corresponding set of positive roots. Given $x \in \mathfrak{g}$, let $Z_{\mathfrak{g}}(x) := \{y \in \mathfrak{g} \mid [yx] = 0\}$ be the centralizer of x in $\mathfrak{g}$. Given $\alpha \in \Phi$, let

$$\mathfrak{m}_\alpha = \mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}.$$

[You may use without proof that $\mathfrak{m}_\alpha$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$.]

(a) Define what it means for $\lambda \in \mathfrak{t}^*$ to be a *dominant weight*. Given a representation V of $\mathfrak{g}$, define what it means for $v \in V$ to be a *highest-weight vector*.

(b) Let V be a finite-dimensional representation of $\mathfrak{g}$. Suppose $v \in V_\lambda$ is a highest-weight vector. Prove that λ is a dominant weight. [You may use any results about the representation theory of $\mathfrak{sl}_2$ and about the root-space decomposition of $\mathfrak{g}$ without proof.]

(c) Now suppose Φ is G_2 and $\alpha \in \Phi$ is a short root. Decompose the adjoint representation of $\mathfrak{g}$ into irreducible $\mathfrak{m}_\alpha$-modules, i.e. find integers $n_1, ..., n_m$ such that $\mathfrak{g} \simeq V(n_1) \oplus ... \oplus V(n_m)$ as a representation of $\mathfrak{m}_\alpha \simeq \mathfrak{sl}_2$. Given a nonzero element $x \in \mathfrak{g}_\alpha$, find $\dim Z_{\mathfrak{g}}(x)$. Briefly explain your logic.

(d) Now Φ is again arbitrary. Show that if $x \in \mathfrak{g}_\alpha$ for some $\alpha \in \Phi$, then $\dim Z_{\mathfrak{g}}(x) \geqslant 3\ell - 2$. [You may use any results from the course.]

3

Let

$$\mathfrak{g} = \mathfrak{so}_6 = \{x \in \mathfrak{gl}_6(\mathbb{C}) \mid xJ + Jx^T = 0\}$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathfrak{t}$ be the space of diagonal matrices in $\mathfrak{g}$, and let Φ be the roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$. For parts (a) - (d), you do not need to provide proofs for your answers.

(a) Explicitly describe the elements of Φ as maps $\mathfrak{t} \to \mathbb{C}$.

(b) Identify a root basis $\Delta \subset \Phi$. Draw and label the Dynkin diagram of $\mathfrak{g}$.

(c) For each $\alpha_i \in \Delta$, explicitly describe the image under the simple reflection w_{α_i} of each element of Δ.

(d) Describe an automorphism of Φ that is not given by an element of the Weyl group.

(e) Briefly explain why $\mathfrak{so}_6 \simeq \mathfrak{sl}_4$. [You may use any result from the course.]

4

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{t}$ and corresponding root system Φ. Let $\Delta = \{\alpha_1, ..., \alpha_\ell\}$ be a choice of root basis and let $\{\omega_1, ..., \omega_\ell\}$ be the fundamental weights with respect to this choice of Δ.

(a) State the Weyl dimension formula, briefly defining the notation you use.

(b) Let $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$, and assume α_1 is a short root. Let $\lambda = a\omega_1 + b\omega_2$ be a dominant weight. Using the Weyl dimension formula, find a formula for $\dim V(\lambda)$ in terms of a and b. [You do not need to prove the Weyl dimension formula.]

(c) Let $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$. Let V be the defining 4-dimensional representation of $\mathfrak{g}$. Find the highest weight of V in terms of ω_1 and ω_2. Decompose $V \otimes V$ into irreducible subrepresentations, i.e. find dominant weights $\lambda_1, ..., \lambda_n$ such that $V \otimes V \simeq V(\lambda_1) \oplus ... \oplus V(\lambda_n)$ as a representation of $\mathfrak{g}$.

5

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{t}$, corresponding root system Φ, Weyl group W, and weight lattice X. Let Δ be a root basis in Φ, and let Φ^+ be the corresponding set of positive roots. Given $\lambda \in X$, let

$$W\lambda = \{w(\lambda) \mid w \in W\}$$

be the orbit of λ under the Weyl group. An irreducible represenation V of $\mathfrak{g}$ is called *minuscule* if there exists $\lambda \in X$ such that if $V_\mu \neq 0$ then $\mu \in W\lambda$. [In the following, you may use any results from the course as long as you state them clearly.]

(a) Suppose $V(\lambda)$ is a minuscule representation of $\mathfrak{g}$ for $\lambda \in X$. Write the formal character of $V(\lambda)$ in terms of $W\lambda$.

(b) Show that if λ is a dominant weight and $V(\lambda)$ is minuscule, then $\langle \lambda, \check{\alpha} \rangle \leqslant 1$ for all $\alpha \in \Phi^+$.

(c) Show that if $X = \mathbb{Z}\Phi$, then $\mathfrak{g}$ has no nontrivial minuscule irreducible representations.

END OF PAPER