UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2018 9:00 am to 12:00 pm

PAPER 101

COMMUTATIVE ALGEBRA

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Throughout this exam, *ring* shall be taken to refer to a commutative ring with identity.

1

Let R and S be rings, and let $\varphi\colon R \to S$ be a ring homomorphism.

(a) Define the *prime spectrum* $\operatorname{Spec}(R)$ and the *Zariski topology* on $\operatorname{Spec}(R)$. Prove that the Zariski topology is in fact a topology.

(b) Show that φ induces a continuous map of topological spaces $f\colon \operatorname{Spec}(S) \to \operatorname{Spec}(R)$.

(c) Prove that $f(\text{Spec(S)})$ is dense in Spec(R) if and only if $\ker(\varphi)$ is contained in the nilradical of R.

(d) Describe the set $\operatorname{Spec}(\mathbb{R}[x])$ and the function $f\colon \operatorname{Spec}(\mathbb{C}[x]) \to \operatorname{Spec}(\mathbb{R}[x])$ induced by the inclusion $\varphi\colon \mathbb{R}[x] \hookrightarrow \mathbb{C}[x]$. [You may use any results about factorization of polynomials in $\mathbb{R}[x]$, provided you state them clearly.]

2

Let R be a ring and $P \subset R$ be a prime ideal.

(a) Define the *localisation* R_P and show that it is an R-algebra.

(b) When is the natural map $R \to R_P$ injective? Justify your answer.

(c) Suppose R_P contains no non-zero nilpotent elements for any prime ideal $P \subset R$. Can R contain non-zero nilpotent elements? Justify your answer. [You must prove any results about local properties that you use.]

(d) Suppose R_P contains no non-zero zero divisors for any prime ideal $P \subset R$. Can R contain non-zero zero divisors? Justify your answer. [You must prove any results about local properties that you use.]

(e) Let R be the infinite direct product $k^{\mathbb{N}}$ of a field k with itself countably many times. Show that R is not Noetherian, but R_P is Noetherian for all prime ideals $P \subset R$.

3 Let R be a ring and $P \subset R$ be a prime ideal.

Define a *P-primary ideal.*

Which of the following statements are true? Give proofs or counterexamples as appropriate.

(a) An intersection of finitely many P-primary ideals is P-primary.

(b) If P is maximal and I is P-primary, then I is a power of P.

(c) Let n be a positive integer. Then P^n is P-primary if and only if $P^n = P^{(n)}$, where $P^{(n)}$ denotes the nth symbolic power of P.

(d) Every irreducible ideal in a Noetherian ring is primary.

(e) Every ideal in a Noetherian ring can be uniquely written as a finite intersection of primary ideals.

4

(a) Define what it means for a ring to be *Noetherian* and *Artinian.*

(b) Let R be a Noetherian ring. Is the power series ring $R[[x]]$ necessarily Noetherian? Justify your answer, proving any results about Noetherian rings that you use.

(c) Let R be a ring and suppose that $R[[x]]$ is Noetherian. Is R necessarily Noetherian? Justify your answer, proving any results about Noetherian rings that you use.

(d) Let R be an Artinian ring. Is $R[[x]]$ necessarily Artinian? Justify your answer.

(e) Let R be a Noetherian ring and S a ring satisfying $R \subseteq S \subseteq R[x]$. Is S necessarily Noetherian? Justify your answer.

5

(a) Define what it means for a ring to be a *Dedekind domain.*

(b) State the *unique factorisation theorem* for ideals in a Dedekind domain.

(c) Prove that any ideal in a Dedekind domain is generated by at most two elements. [You must clearly state any results that you use.]

(d) Give an example of a Dedekind domain that is not a principal ideal domain. Justify your answer.

6

(a) Define the *Krull dimension* of a ring.

(b) What is the Krull dimension of a discrete valuation ring? Justify your answer.

(c) Let R be a ring and $S \subseteq R$ a subring. Define what it means for R to be *integral* over S.

(d) Let R be integral over S. Prove that R and S have the same Krull dimension. [You must state clearly any results that you use.]

(e) Consider the integral extension $\mathbb{Z} \subset \mathbb{Z}[i]$. Let P be a prime ideal of $\mathbb{Z}[i]$ and let $Q = P \cap \mathbb{Z}$ denote the corresponding prime ideal of $\mathbb{Z}$. Is the localisation $\mathbb{Z}[i]_P$ necessarily integral over $\mathbb{Z}_Q$? Justify your answer.

END OF PAPER