UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 9 June, 2017 1:30 pm to 3:30 pm

PAPER 345

ENVIRONMENTAL FLUID DYNAMICS

Attempt no more than ***THREE*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A source of buoyancy with buoyancy flux B and volume flux Q is supplied to the floor of an enclosed space with cross-sectional area A and height H. If the buoyancy flux leads to formation of a turbulent buoyant plume, derive an expression for the speed of the first-front descending from the top of the space. You may assume the plume behaves as a pure plume, for which the volume flux at a distance z above the point source is $\lambda B^{1/3} z^{5/3}$.

If air is pumped into the room at low level and removed from the ceiling of the room, with a total volume flux $Q_v << \lambda B^{1/3} H^{5/3}$, show that a two layer stratification develops. Find an expression for the height of the interface and the buoyancy of the fluid above the interface.

The source of buoyancy is now divided amongst n equal sources, and thereby produces n independent plumes, each with buoyancy flux B/n and volume flux Q_v/n at the source. If the air is pumped into the room at low level through these sources and removed from the ceiling of the room with total flow rate Q_v, explain why a two layer stratification again becomes established and find the height of the interface and the buoyancy of the upper layer in this case.

If the background flow is adjusted to value $Q_v/2$, using dimensional analysis or otherwise develop an estimate for the time required for the system to return to equilibrium.

2

A finite volume of contaminant, V_c per unit length, is suddenly released along a line source which supplies a steady state two-dimensional turbulent buoyant plume with buoyancy flux B per unit length. By writing down the conservation laws for a line plume, show that the volume flux in the plume at height z above the source scales as $V = \lambda B^{1/3} z$ and that the width of the plume scales as $w = \mu z$ where λ and μ are constants which should be related to the entrainment coefficient.

Show also that the horizontal integral of concentration c evolves with distance z above the source as

$$\frac{\partial c}{\partial t} + \alpha B^{1/3} \frac{\partial c}{\partial z} = \beta B^{1/3} \frac{\partial}{\partial z} \left[z \frac{\partial c}{\partial z} \right]$$

where α and β are constants. Derive a solution for the horizontally averaged concentration as a function of the distance from the source z and the time after release, t. Show that according to this solution, the maximum concentration occurs at a distance $z = \alpha B^{1/3} t$ ahead of the source.

3

A turbulent gravity current of volume flux $Q = uhw$, where h is the depth and u the speed of the flow, propagates along a channel of width w. The current contains particles with fall speed v_s and concentration c, and the buoyancy may be expressed in terms of this concentration as $g' = g_o c$. Show that the evolution of the current may be described in terms of the shallow water equations

$$\frac{\partial h}{\partial t} + \frac{\partial uh}{\partial x} = 0$$

$$\frac{\partial hg'}{\partial t} + \frac{\partial ug'h}{\partial x} = -v_s g'$$

$$\frac{\partial hu}{\partial t} + \frac{\partial u^2 h}{\partial x} = -\frac{1}{2}\frac{\partial h^2 g'}{\partial x}$$

where x is the along-channel position. In the case of a steady current, derive an expression for the rate of change of the buoyancy as a function of position along the channel. You may assume that if a particle settles from the flow it is not re-entrained. Show also that the quantity $u^2 h + \frac{1}{2}h^2 g'$ is independent of position and hence deduce how the depth of current evolves with distance downstream.

Generalise the model to account for the case in which there are two populations of particles, with corresponding buoyancies g_1' and g_2' and with fall speed v_1 and v_2. In this case derive an expression for the fraction of particles with fall speed v_1 in the deposit as a function of the distance from the source.

END OF PAPER