UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 9 June, 2017 9:00 am to 11:00 am

PAPER 344

THEORETICAL PHYSICS OF SOFT CONDENSED MATTER

*You may attempt **ALL** questions, although full marks can be achieved by good answers to **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Answer all parts of the question.

(a) Explain why the coarse-grained dynamics of an equilibrium system with microscopic time-reversal symmetry must obey

$$\frac{\mathbb{P}_F}{\mathbb{F}_B} = \exp[-\beta(F_2 - F_1)]$$

where $\mathbb{P}_{F,B}$ are forward and backward path probabilities, $\beta = 1/k_BT$, and F_2 and F_1 are the final and initial free energies.

(b) A certain system with a non-conserved scalar order parameter $\phi(\mathbf{r}, t)$ is described by the stochastic field equation

$$\dot{\phi}(\mathbf{r}, t) = -\Gamma \frac{\delta F}{\delta \phi(\mathbf{r})} + \eta(\mathbf{r}, t)$$

where Γ is a constant and η is a Gaussian process that obeys $\langle \eta(\mathbf{r}, t)\eta(\mathbf{r}', t')\rangle = \sigma^2 \delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$. Show, using the result of part (a) or otherwise, that $\sigma^2 = 2k_BT\Gamma$.

(c) Consider the case where $F = \int \mathbb{F}(\phi, \nabla\phi)d\mathbf{r}$ with $\mathbb{F} = \frac{a}{2}\phi^2 + \frac{\kappa}{2}(\nabla\phi)^2$ and $a > 0$. Give the equation of motion for a Fourier component $\phi_{\mathbf{q}}(t)$ of the order parameter, and show that this is solved by

$$\phi_{\mathbf{q}}(t) = \phi_{\mathbf{q}}(0) \exp[-r(q)t] + \int_0^t \eta_{\mathbf{q}}(t') \exp[-(t - t')r(q)]\, dt'$$

where you should give an expression for the decay rate $r(q)$, but are not asked to prove that $\langle \eta_{\mathbf{q}}(t)\eta_{\mathbf{q}'}, t')\rangle = 2k_BT\Gamma\delta_{\mathbf{q},-\mathbf{q}'}\delta(t - t')$.

(d) Consider such a system that is prepared in an equilibrium state with $a = a_0$ at time $t = 0$. Suppose conditions are then suddenly altered so that $a = a_1$ for $t > 0$. (Both a_0 and a_1 are positive.) Show that the equal-time correlator $S_q(t) = \langle |\phi_{\mathbf{q}}(t)|^2\rangle$ obeys

$$S_q(t) = S_q(0) \exp[-2r(q)t] + \frac{k_BT}{a_1 + \kappa q^2}\big(1 - \exp[-2r(q)t]\big)$$

where again you should give an expression for $r(q)$.

(e) Suppose that instead the system is prepared with an initial condition comprising a localized density peak: $\phi(\mathbf{r}, 0) = A\delta(\mathbf{r} - \mathbf{r}')$. Without detailed derivation, describe the resulting time evolution.

2

Answer all parts of the question.

For an incompressible binary fluid mixture, the hydrodynamic-level equations read

$$\begin{aligned} (\partial_t + \mathbf{v}.\nabla)\phi &= -\nabla.\mathbf{J} && (1)\\ \mathbf{J} &= -M\nabla\mu && (2)\\ \rho(\partial_t + \mathbf{v}.\nabla)\mathbf{v} &= \eta\nabla^2\mathbf{v} - \nabla P - \phi\nabla\mu && (3)\\ \nabla.\mathbf{v} &= 0 && (4)\\ \mu &= a\phi + b\phi^3 - \kappa\nabla^2\phi && (5) \end{aligned}$$

(a) Without deriving them, briefly outline why the equations have this form.

(b) In the late-stage coarsening of a bicontinuous fluid mixture in three dimensions, it is argued that (3) can be schematically represented for scaling purposes as

$$\rho(\alpha\ddot{L} + \beta\dot{L}^2/L) = \eta\gamma\dot{L}/L^2 + \sigma\delta/L^2 \tag{6}$$

with $L(t)$ the characteristic domain size, $\sigma(a, b, \kappa)$ an interfacial tension, and $\alpha, \beta, \gamma, \delta$ dimensionless quantities of order unity. What are the assumptions behind (6)? Noting that the only combinations of parameters ρ, σ, η with units of length and of time are respectively $L_0 = \eta^2/\rho\sigma$ and $t_0 = \eta^3/\rho\sigma^2$, deduce that $L(t)/L_0 = f(t/t_0)$ and find a nondimensionalised version of (6) satisfied by the function $f(u)$.

(c) Find power-law scalings for $f(u)$ that cause $L(t)$ to become independent of (i) ρ and (ii) η, and show that the respective 'viscous hydrodynamic' and 'inertial hydrodynamic' scalings capture the primary balance of terms in (6) at (i) small u and (ii) large u.

(d) Consider now the late-stage coarsening of a bicontinuous fluid mixture that takes place within a three dimensional microporous medium (such as a polymer network). This can be modelled by replacing the viscous term in (3) with a local drag term on the fluid:

$$\rho(\partial_t + \mathbf{v}.\nabla)\mathbf{v} = -\bar{\eta}\mathbf{v} - \nabla P - \phi\nabla\mu \tag{7}$$

Show that now $L(t)/L_1 = g(t/t_1)$ where $L_1^3 = \sigma\rho/\bar{\eta}^2$ and $t_1 = \rho/\bar{\eta}$.

(e) Show that if $g(u) \sim u^y$ with $y > 0$, the inertial terms are always negligible compared to the local drag term at large u. Find the power-law scaling that arises in the long time limit in this model.

3

Answer all parts of the question.

In a certain 2D nematic, the local free energy density is (in the "one elastic constant" approximation)

$$\mathbb{F} = a\mathrm{Tr}(\mathbf{Q}^2) + b(\mathrm{Tr}(\mathbf{Q}^2))^2 + \frac{K}{2}|\nabla.\mathbf{Q}|^2$$

where b and K are positive constants, and $\nabla.\mathbf{Q} \equiv \nabla_i Q_{ij}$.

(a) Explain why the order parameter for a nematic is a traceless symmetric tensor $\mathbf{Q}$. Show that for $a < 0$ the free energy is minimized by a uniform order parameter field $Q_{ij} = \lambda_0[\hat{n}_i\hat{n}_j - \delta_{ij}/2]$, where $\mathbf{n}$ is a constant unit vector, and find λ_0 in terms of a and b.

(b) In a certain 'vortex' configuration of topological charge $q = +1$, the director $\mathbf{n}(\mathbf{r})$ at radius r from a fixed origin $\mathbf{r} = \mathbf{0}$ lies tangential to the circle at that radius, so that

$$Q_{ij}(\mathbf{r}) = \lambda(r)[\theta_i\theta_j - \delta_{ij}/2]$$

where $\mathbf{r} \equiv r(\cos\theta, \sin\theta)$; unit vector $\boldsymbol{\theta} \equiv (-\sin\theta, \cos\theta)$, and $\lambda(r)$ is the local strength of the nematic ordering. Show that the local free energy density can then be written

$$\mathbb{F} = \frac{a}{2}\lambda^2 + \frac{b}{4}\lambda^4 + \frac{K}{2}\left|((\nabla\lambda).\boldsymbol{\theta})\boldsymbol{\theta} + \lambda(\nabla.\boldsymbol{\theta})\boldsymbol{\theta} + \lambda(\boldsymbol{\theta}.\nabla)\boldsymbol{\theta} - \frac{\nabla\lambda}{2}\right|^2$$

(c) Noting that λ differs from λ_0 only in a local region near the vortex core (of size r_0), establish that the free energy ΔF of the vortex, relative to a defect-free state, is dominated by a term $\propto K\lambda_0^2\ln(L/r_0)$. What determines the large-scale cutoff, L?

(d) Is this defect stable (i) topologically? (ii) energetically?

(e) Suppose a 2D nematic is prepared in an initial state with a vortex of topological charge $+1$ separated by a large distance from an 'antivortex' of topological charge -1. Describe what you expect to happen en route to a final, defect-free state.

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