UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2017 9:00 am to 11:00 am

PAPER 343

QUANTUM FLUIDS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Dynamics of a Bose-Einstein condensate is described by the Gross-Pitaevskii equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + g|\psi|^2\psi - \mu\psi, \qquad (*)$$

where $\hbar$ is the Plank's constant, m is the particle mass, μ is the chemical potential, and g is the effective particle interaction. The number density is $n(\mathbf{x}) = |\psi(\mathbf{x})|^2$.

(i) Write down the Hamiltonian functional $H[\psi, \psi^*]$ for the Gross-Pitaevskii equation and find the number density for the ground state of the condensate.

(ii) Show that a dimensionless form of (*) can be written as

$$-2i\frac{\partial\tilde{\psi}}{\partial\tilde{t}} = \tilde{\nabla}^2\tilde{\psi} + (1 - |\tilde{\psi}|^2)\tilde{\psi}, \qquad (**)$$

where you need to identify dimensionless variables used.

(iii) Assume that there is a solid wall at $y = 0$ in a two-dimentional condensate. Show that the number density of the condensate is given by $n(y) = \tanh^2(y/\xi)$, where you need to determine ξ.

(iv) Write down the equation and the boundary condition for two-dimensional solitary waves moving with velocity v in the positive x-direction in a condensate described by (**) in the frame of reference in which the solitary wave is stationary. Write down the convergent integrals for the energy E and momentum $\mathbf{p}$ of the solitary wave. Show that along the sequence of the solitary waves $v = \partial E/\partial p_x$, where p_x is the momentum in x direction.

2

Consider a cylindrical bucket of height H_0 and radius R_0 containing a Bose-Einstein condensate subject to an external harmonic trap $V_{\text{ext}}(r) = m\omega_r^2 r^2/2$ perpendicular to the axis of the cylinder, where m is atomic mass, ω_r is the trap frequency.

(i) Find the Thomas-Fermi profile of the condensate, $n(\mathbf{r})$ and write the expression for the radial Thomas-Fermi radius, R_r, in terms of the density along the axis, n_0, the interaction strength, g, and ω_r.

(ii) Estimate the kinetic energy E_{kin} due to a vortex along the cylinder axis using

$$E_{\text{kin}} = \int_V \frac{1}{2} m n(\mathbf{r}) u_\theta^2(\mathbf{r})\, d\mathbf{r},$$

where V is the volume of the cylindrical shell of the height H_0, with the inner radius given by the vortex core size a_0 and the outer radius by R_r. Your answer should be expressed in terms of H_0, n_0, R_r and a_0.

(iii) Estimate the angular momentum, L_z, of the vortex state in the same volume V.

(iv) Use the fact that the vortex core size, $a_0 \ll R_r$ to drop the terms quadratic in a_0 in E_{kin} and L_z and hence estimate the critical rotation frequency at which the presence of a vortex becomes energetically favourable in comparison with the vortex-free state.

3

Exciton-polariton condensates are often modelled by the complex Ginzburg-Landau equation (cGLE) written for the condensate wavefunction, $\psi(\mathbf{r}, t)$, and the rate equation on the hot exciton reservoir, $\mathcal{R}(\mathbf{r}, t)$:

$$i\frac{\partial \Psi}{\partial t} = -\nabla^2\Psi + U_0|\Psi|^2\Psi + g_R\mathcal{R}\Psi + \frac{i}{2}\Big(R_R\mathcal{R} - \gamma_C\Big)\Psi, \quad (1)$$

$$\frac{\partial \mathcal{R}}{\partial t} = -\left(\gamma_R + R_R|\Psi|^2\right)\mathcal{R} + P(\mathbf{r}), \quad (2)$$

where U_0 and g_R are the strengths of effective polariton-polariton interactions and the blue-shift due to the interactions with non-condensed particles, respectively, R_R is the rate at which the exciton reservoir feeds the condensate, and P is the pumping into the exciton reservoir. Finally, γ_C is the rate of losses of condensed polaritons through the cavity mirrors and γ_R is the rate of redistribution of reservoir excitons between the different energy levels.

(i) Show that under suitable assumptions, that you need to clearly state, Eqs. (1) and (2) reduce to a single cGLE

$$\frac{\partial \Psi}{\partial t} = (\alpha - \beta|\Psi|^2)\Psi + i(\nabla^2 - g|\Psi|^2 + s)\Psi, \quad (3)$$

where you need to write the expressions for α, β, g and s.

(ii) Assume the spatially uniform pumping and find the uniform solution of Eq. (3) stating the corresponding number density, n_∞, and the chemical potential, μ.

Show that to find a stationary solution one would need to solve

$$[\nabla^2 + \xi(1 - |\psi|^2)]\psi = 0, \quad \psi(\infty) \to 1, \quad (4)$$

where ξ is a complex parameter that you need to specify. How does ψ relate to Ψ?

(iii) Use Eq. (4) to find the ordinary differential equations that describe the amplitude and the radial component of the velocity of a straight line vortex with multiplicity 1. What is the slope of the radial velocity at the vortex centre?

END OF PAPER