UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 5 June, 2017 1:30 pm to 4:30 pm

PAPER 342

BIOLOGICAL PHYSICS AND COMPLEX FLUIDS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

An elastic filament of length L, radius a, density ρ_f, and elastic modulus A is immersed in a fluid of viscosity μ and density ρ. The filament extends along the positive x-axis and its left end is clamped at the origin, while its right end is free. Under the action of gravity, and assuming $\rho_f > \rho$, the filament will bend downwards.

(a) Write down the energy functional for the filament, including the contributions of bending elasticity and the gravitational potential energy.

(b) Using the ideas of resistive force theory, find the equation of motion of the filament for small-amplitude deformations $h(x,t)$ in terms of $\zeta_\perp$, the drag coefficient for motion perpendicular to the filament's long axis. State the boundary conditions that hold at $x = 0$ and $x = L$. Solve for the steady-state profile after transients have died away. From this solution, explain how the filament acts as a Hookean spring under the total gravitational force on the filament, and using the displacement at the free end as a measure of the deflection, calculate its effective spring constant. Show consistency using dimensional analysis.

(c) If gravity is now turned off, the filament relaxes back to the x-axis. Find the dominant long-time behaviour of $h(x,t)$ as it approaches 0.

2

The constitutive relationship for a non-Newtonian fluid, termed the co-rotational Maxwell fluid, relates the deviatoric stress in the fluid, $\boldsymbol{\sigma}$, to the shear rate tensor, $\dot{\boldsymbol{\gamma}}$, as

$$\boldsymbol{\sigma} + \lambda \frac{\mathcal{D}\boldsymbol{\sigma}}{\mathcal{D}t} = \mu \dot{\boldsymbol{\gamma}}, \qquad (\dagger)$$

where λ and μ are constants and where the objective co-rotational derivative, $\frac{\mathcal{D}}{\mathcal{D}t}$, is defined for a tensor $\mathbf{a}$ as

$$\frac{\mathcal{D}\mathbf{a}}{\mathcal{D}t} = \frac{\partial \mathbf{a}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{a} + \frac{1}{2} \left(\boldsymbol{\omega} \cdot \mathbf{a} - \mathbf{a} \cdot \boldsymbol{\omega} \right),$$

where the tensor $\boldsymbol{\omega}$ is twice the antisymmetric part of the velocity gradient tensor, $\boldsymbol{\omega} = \nabla \mathbf{u} - \nabla \mathbf{u}^T$.

(a) What is meant by the statement that $\frac{\mathcal{D}}{\mathcal{D}t}$ is *objective*? Give another example of an objective derivative and give an example that is not objective.

(b) Under small deformations, the nonlinear constitutive relationship in Eq. $(\dagger)$ maybe be linearised. What is the name of the flow described by the linearised version of Eq. $(\dagger)$? What are the physical interpretations of μ and λ for that linearised fluid? Explain briefly why that linearised fluid has no normal stress differences.

(c) The fluid characterised by the full nonlinear constitutive relationship in Eq. $(\dagger)$ undergoes steady two-dimensional shear with shear rate $\dot{\gamma}$. Determine its steady shear viscosity and show that it is shear-thinning. Determine the two normal stress differences. Are their signs and relative magnitudes consistent with experiments?

3

Experiments by Adler have shown that a cloud of motile bacteria can move steadily down a liquid-filled capillary tube as they consume nutrient. Keller and Segel proposed the following model for this process, in which nutrient diffusion is neglected,

$$\frac{\partial b}{\partial t} = D\frac{\partial^2 b}{\partial x^2} - \frac{\partial}{\partial x}\left(\chi b \frac{\partial c}{\partial x}\right), \tag{1}$$

$$\frac{\partial c}{\partial t} = -kb \tag{2}$$

where the bacterial and nutrient densities are $b(x,t)$ and $c(x,t)$, respectively, and the consumption rate k and diffusion constant D are assumed constant. The chemotactic response coefficient χ is taken to have the singular form

$$\chi = \frac{\alpha}{c},$$

where α is a constant.

(a) Assume a solution for b and c in the form of a rightward traveling wave, with as yet unknown speed v, and find the equations for the associated traveling-wave functions $B(z)$ and $C(z)$, with $z = x - vt$.

(b) Solve the equations in (a) and show that if $\alpha > D$,

$$\lim_{z\to\infty} C = C_\infty, \quad \lim_{z\to\infty} B = \lim_{z\to-\infty} C = \lim_{z\to-\infty} B = 0.$$

Show that by suitable choice of an integration constant that

$$\frac{C}{C_\infty} = \left(1 + \mathrm{e}^{-\xi}\right)^{-1/(\mu-1)},$$

where $\xi = vz/D$, and $\mu = \alpha/D$. Show that C monotonically increases, whereas B has a single maximum. Sketch graphs of these functions. What is the biological meaning of C_∞?

(c) From the governing traveling-wave equations show that the speed v satisifes

$$v = Nk/aC_\infty,$$

where N is the total number of bacteria in the tube (assumed infinity long) and a is its cross-sectional area. Comment on the depedence of v on the parameters.

4

(a) Consider two incompressible solutions of Stokes equations with velocities $\mathbf{u}$ and $\hat{\mathbf{u}}$, stress fields $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$ and no body forces. These solutions occupy the same fluid volume V bounded by the same surface S with normal $\mathbf{n}$ into the fluid. Prove the reciprocal theorem for Stokes flows i.e.

$$\iint_S \hat{\mathbf{u}} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}\, \mathbf{d}S = \iint_S \mathbf{u} \cdot \hat{\boldsymbol{\sigma}} \cdot \mathbf{n}\, \mathbf{d}S.$$

(b) A spherical particle of radius a rotates with angular velocity $\boldsymbol{\Omega}$ by instantaneously imposing a velocity distribution $\mathbf{u}'$ along its spherical boundary (measured in the particle frame) in an infinite fluid. By applying the reciprocal theorem to the free-rotating problem (flow $\mathbf{u}$) and to the problem of solid-body rotation at angular velocity $\hat{\boldsymbol{\Omega}}$ (flow $\hat{\mathbf{u}}$, for which you are given that $\hat{\boldsymbol{\sigma}} \cdot \mathbf{n} = -3\mu\hat{\boldsymbol{\Omega}} \times \mathbf{n}$ on the sphere), show that

$$\boldsymbol{\Omega} = -\frac{3}{8\pi a^3} \int_S \mathbf{n} \times \mathbf{u}' dS \qquad (\dagger\dagger).$$

(c) The spherical particle has a prescribed value of $\mathbf{u}'$ along its boundary,

$$\mathbf{u}' = \sin\theta \left(\sum_{n \geqslant 1} a_n \sin n\phi + \sum_{n \geqslant 0} b_n \cos n\phi \right) \mathbf{e}_\phi,$$

where a_n and b_n are constants, and where the angle ϕ denotes the azimuthal angle (ranging from 0 to 2π) and θ the polar angle (ranging from 0 to π) in spherical coordinates. Determine all components of $\boldsymbol{\Omega}$. Show that is is possible to pick the values of a_n and b_n so that the particle rotates but creates no flow disturbance.

END OF PAPER