UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2017 9:00 am to 12:00 pm

PAPER 341

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

*Attempt **THREE** questions from Section A and **ONE** question from Section B.*

*There are **SEVEN** questions in total.*

Each question in Section B carries twice the weight of a question from Section A.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

(i) Prove that the Runge–Kutta method with the Butcher tableau is a collocation method and determine its order.

$$\begin{array}{c|cc} \frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{3}{4} & \frac{1}{4} \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array}$$

(ii) Is the method A-stable?

2

Let $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, $\mathbf{y}(0) = \mathbf{y}_0$, be an initial problem ODE. We assume that the function $\mathbf{f}$ is as smooth as necessary and let

$$\mathbf{g}(\mathbf{y}) = \frac{\partial \mathbf{f}(\mathbf{y})}{\partial \mathbf{y}} \mathbf{f}(\mathbf{y})$$

– hence $\mathbf{y}'' = \mathbf{g}(\mathbf{y})$. Consider the two-step, two-derivative method

$$\mathbf{y}_{n+2} - (1+\alpha)\mathbf{y}_{n+1} + \alpha \mathbf{y}_n = (1-\alpha) h \mathbf{f}(\mathbf{y}_{n+2}) - \frac{1}{2} h^2 (1-3\alpha) \mathbf{g}(\mathbf{y}_{n+2}).$$

(i) Determine the formal order of the method for different values of the real parameter α.

(ii) For which values of α is the method convergent?

(iii) Check A-stability for the following two values: $\alpha = \frac{1}{7}$ and $\alpha = \frac{1}{2}$.

[You may assume without proof that the Dahlquist Equivalence Theorem is valid in this setting.]

3

(i) Consider the PDE

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \alpha u, \qquad -1 \leqslant x \leqslant 1, \quad t \geqslant 0,$$

given with zero boundary conditions at $x = \pm 1$ and appropriate initial conditions at $t = 0$. The constant α is real.

Determine the conditions on α so that every solution of this PDE is uniformly bounded for $t \geqslant 0$ and $-1 \leqslant x \leqslant 1$.

(ii) The PDE is semi-discretized by the ODE system

$$u_m'' = \frac{1}{\Delta x}(u_{m-1} - 2u_m + u_{m+1}) + \alpha u_m,$$

where $u_m \approx u(m\Delta x)$, $m = 1, \ldots, M$, and $\Delta x = 1/(M+1)$. Is the method stable?

4

The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad -1 \leqslant x \leqslant 1, \quad t \geqslant 0,$$

given with an initial condition at $t = 0$ and zero Dirichlet boundary conditions, is solved by the two-step method

$$u_{m+2}^{n+2} - \frac{4}{3}u_m^{n+1} + \frac{1}{3}u_m^n = \frac{2}{3}\mu(u_{m-1}^{n+2} - 2u_m^{n+2} + u_{m+1}^{n+2}),$$

where $u_m^n \approx u(m\Delta x, n\Delta t)$ and $\mu = \Delta t/(\Delta x)^2$ is the Courant number.

(i) Determine the order of the method.

(ii) Find the range of $\mu > 0$ for which the method is stable.

5

Let $f \in L_2[0,1]$. The Airy-type equation

$$-y'' + xy = f, \qquad 0 \leqslant x \leqslant 1,$$

given with zero boundary conditions at $x = 0$ and $x = 1$, is solved by the Ritz method.

(i) Quoting all the appropriate definitions and theorems, find the underlying variational problem.

(ii) Derive the set of linear algebraic equations that occur once the above variational problem is projected to a finite-dimensional space.

(iii) The finite-dimensional space in question being a linear combination of hat functions over uniform grid, find the equations explicitly.

SECTION B

6

Write an essay about linear and nonlinear stability analysis of Runge–Kutta methods. Your essay should include proofs and be accompanied by examples.

7

Write an essay, inclusive of examples, on techniques to determine stability of fully-discretized initial-value problems for PDEs.

END OF PAPER