UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 7 June, 2017 9:00 am to 12:00 pm

PAPER 340

TOPICS IN MATHEMATICS OF INFORMATION

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Give the definition of a *multiresolution analysis* (MRA) for $L^2(\mathbb{R})$. Define the low pass filter m associated with the scaling function φ of this MRA.

(b) Explain (without proof) how one can construct an orthonormal wavelet from an MRA. Write down a formula for an orthonormal wavelet in terms of the scaling function and the low pass filter of the associated MRA.

(c) Suppose that the scaling function φ is an element of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Show that if the associated wavelet ψ has p vanishing moments, then the derivatives of the low pass filter m satisfy

$$m^{(k)}(\pi) = 0, \qquad k = 0, \ldots, p-1.$$

State clearly all theorems (without proof) that you use.

(d) Hence deduce that the derivatives of the Fourier transform of φ satisfy for $k = 0, \ldots, p-1$,

$$\hat{\varphi}^{(k)}(2\pi j) = 0, \qquad \forall j \in \mathbb{Z} \setminus \{0\}.$$

.

2

(a) Let $\mathcal{B} := \{g_m\}_{m\in\mathbb{N}}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. Given $f \in \mathcal{H}$, define its *N-term linear approximation* f_N^{lin} and its *N-term nonlinear approximation* f_N^{nonlin} with respect to $\mathcal{B}$.

(b) For $\alpha > 0$, define what it means for a function, f, to be uniformly Lipschitz-α over $[0, 1]$. Define the space C^α of Hölder-α functions.

In the following, let $\mathcal{H} = L^2([0, 1])$ and let $\{\psi_{j,l}\}_{j,l}$ be an orthonormal basis generated by a Daubechies wavelet which is q times continuously differentiable and has q-vanishing moments, and adapted to the interval $[0, 1]$.

(c) Given a function $f \in C^\alpha$ with $0 < \alpha < q$, show that $|\langle f, \psi_{j,n}\rangle| \leqslant C2^{-j(\alpha+1/2)}\|f\|_{C^\alpha}$ where the constant $C > 0$ is independent of f.

(d) Suppose that $f \in L^\infty([0, 1])$ has K discontinuities and is uniformly Lipschitz-α between these discontinuities, where $1/2 < \alpha < q$. Show that the nonlinear approximation error $\epsilon_n(N, f) := \|f_N^{nonlin} - f\|^2_{L^2([0,1])}$ satisfies

$$\epsilon_n(N, f) = \mathcal{O}(N^{-2\alpha}).$$

You may use without proof the fact that there exists a constant $C > 0$ such that for all $g \in L^\infty([0, 1])$,

$$|\langle\psi_{j,n}, g\rangle| \leqslant C2^{-j/2}\|g\|_{L^\infty([0,1])}.$$

3

(a) Let $A \in \mathbb{C}^{m \times N}$ and let $\mathcal{N}(A)$ denote the null space of A. What does it mean for A to satisfy the null space property of order s? Assuming that A satisfies the null space property of order s, show that $\mathcal{N}(A)$ does not contain any $2s$-sparse vector other than the zero vector. Hence deduce that given any s-sparse vector, which we denote by x, x is the unique solution to

$$\min_{z \in \mathbb{C}^N} \|z\|_0 \text{ subject to } Ax = Az.$$

Let $y \in \mathbb{C}^m$ and let $q \in (0, 1]$. Consider the minimization problem

$$\min_{z \in \mathbb{C}^N} \|z\|_q \text{ subject to } Az = y. \qquad (P_q)$$

(b) Show that any s-sparse vector x is the unique minimizer to (P_q) with $y = Ax$ if and only if for all $S \subset \{1, \dots, N\}$ with $|S| \leqslant s$,

$$\|v_S\|_q^q < \|v_{S^c}\|_q^q, \qquad \forall v \in \mathcal{N}(A) \setminus \{0\}.$$

In the above, $S^c := \{1, \dots, N\} \setminus S$. You may use the fact that $(a + b)^q \leqslant a^q + b^q$ for all $a, b \geqslant 0$ and $q \in (0, 1)$.

(c) Show that if any s-sparse vector x is the unique minimizer to (P_q) with $y = Ax$, then any s-sparse vector x is the unique minimizer to (P_p) with $y = Ax$ and $p \in (0, q)$.

4

(a) Let $\Omega = (0,1)^2$ and let $u \in L^1(\Omega)$. State the definition of the total variation of u and define the space $BV(\Omega)$ with the corresponding norm.

(b) Give an example of a function which is in $BV(\Omega)$ but not in the Sobolev space $W^{1,1}(\Omega)$. Briefly justify your answer (You may state without proof any properties of Sobolev functions that you use).

(c) Let $\alpha > 0$, $\Omega = \mathbb{R}^2$ and let $J : L^2(\Omega) \to [0, \infty]$ be defined by $J(u) = |Du|(\Omega)$, with $J(u) = +\infty$ if $u \notin BV(\Omega)$. Show that u is the minimizer of

$$\min_{v \in L^2(\Omega)} \{\alpha J(v) + \frac{1}{2}\|v - g\|_{L^2}^2\} \tag{1}$$

if and only if

$$\frac{g - u}{\alpha} \in \partial J(u).$$

(d) Let $g(x) = \chi_{B(0,R)}$ be the characteristic function on the ball in $\mathbb{R}^2$ of radius R, centred at 0. Derive an explicit formula for the minimizer u of (1).

5

Let $X = \mathbb{R}^{N\times N}$ for some $N \in \mathbb{N}$ and write $X^2 := X \times X$. For $g \in X$, $\lambda > 0$, consider

$$\min_{u\in X}\{\lambda\|\nabla u\|_{2,1} + \frac{1}{2}\|u-g\|_2^2\}$$

where $\|q\|_2^2 = \sum_{i,j} q_{i,j}^2$ for $q \in X$, $\|p\|_{2,1} = \sum_{i,j}\sqrt{(p_{i,j}^1)^2 + (p_{i,j}^2)^2}$ for $p = (p^1, p^2) \in X^2$, and $\nabla : X \to X^2$ is the discrete gradient operator defined by

$$(\nabla u)_{i,j} = \begin{pmatrix}(D_x^+ u)_{i,j}\\(D_y^+ u)_{i,j}\end{pmatrix},$$

with

$$(D_x^+ u)_{i,j} = \begin{cases} u_{i+1,j} - u_{i,j} & i < N\\ 0 & i = N,\end{cases} \qquad (D_y^+ u)_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j} & j < N\\ 0 & j = N.\end{cases}$$

This question is about the dual formulation of this minimization problem and the use of the projected gradient descent algorithm.

(a) Find an expression for the minimizer u as the projection of g onto a closed convex set. Carefully justify every step of your derivation and quote any definitions and theorems that you use.

(b) Give an iterative algorithm to compute this projection and state under which conditions the iterates converge.

(c) Briefly describe the convex set associated with the projection when you consider instead for $q \geqslant 1$, the minimization problem

$$\min_{u\in X}\{\lambda\|Au\|_{q,1} + \frac{1}{2}\|u-g\|_2^2\}$$

where $A : X \to X^d$ for $d \in \mathbb{N}$ is a linear operator and $\|v\|_{q,1} = \sum_{i,j}\left((v_{i,j}^1)^q + \cdots (v_{i,j}^d)^q\right)^{1/q}$ for $v = (v^1, \cdots, v^d) \in X^d$.

END OF PAPER