UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2017 1:30 pm to 3:30 pm

PAPER 339

TOPICS IN CONVEX OPTIMISATION

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $\mathbf{S}^n$ be the space of real symmetric $n \times n$ matrices. A matrix $A \in \mathbf{S}^n$ is called *copositive* if

$$x^T A x \geqslant 0 \quad \forall x \in \mathbb{R}^n_+$$

where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geqslant 0 \text{ for all } i = 1, \ldots, n\}$.

(a) Show that the 2×2 matrix $\left[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right]$ is copositive but not positive semidefinite.

(b) Show that if A can be written as $A = P + N$ where P is positive semidefinite and N is entrywise non-negative (i.e., $N_{ij} \geqslant 0$ for all $1 \leqslant i, j \leqslant n$) then A is copositive.

(c) Let $K \subset \mathbf{S}^n$ be the set of copositive matrices. Show that K is a closed convex cone.

(d) Recall the definition of *pointed cone*. Show that K is a pointed cone and that its interior is not empty.

(e) Recall the definition of *dual cone*. Show that the dual cone of K is $K^* = C$ with

$$C = \text{cl cone}\left\{xx^T : x \in \mathbb{R}^n_+\right\} \tag{1}$$

where cl denotes closure, and cone denotes the conic hull. You can use the following version of the separating hyperplane theorem without proof:

> *Separating hyperplane theorem:* If C is a closed convex cone and $z \notin C$ then there exists y such that $\langle y, z \rangle < 0$ and $\langle y, x \rangle \geqslant 0$ for all $x \in C$.

(f) Show that the closure operation in (1) is not needed, i.e., that $\text{cone}\{xx^T : x \in \mathbb{R}^n_+\}$ is closed. For this you can use Carathéodory's theorem without proof:

> *Carathéodory's theorem:* If $S \subset \mathbb{R}^N$ then any element of cone(S) is a conic combination of at most $N = \dim(\mathbb{R}^N)$ elements of S.

(g) Let $Q \in \mathbf{S}^n$ and consider the quadratic optimisation problem

$$\underset{x \in \mathbb{R}^n}{\text{minimise}} \quad x^T Q x \quad \text{subject to} \quad \sum_{i=1}^n x_i^2 = 1 \text{ and } x_i \geqslant 0 \; \forall i = 1, \ldots, n. \tag{2}$$

Show that (2) has the same optimal value as

$$\underset{\lambda \in \mathbb{R}}{\text{maximise}} \quad \lambda \quad \text{subject to} \quad Q - \lambda I \in K. \tag{3}$$

where I is the $n \times n$ identity matrix.

(h) Write the dual of (3) as a conic program over $K^* = C$.

2

Let $S \in \mathbb{R}^{n\times m}$ and consider the following binary quadratic optimisation problem:

$$\text{maximise} \quad p^T S q \quad \text{subject to} \quad p \in \{-1,1\}^n, q \in \{-1,1\}^m.$$

(a) Show that this problem has the same optimal value as

$$\text{maximise} \quad x^T A x \quad \text{subject to} \quad x \in \{-1,1\}^{n+m} \tag{1}$$

where

$$A = \frac{1}{2}\begin{bmatrix} 0 & S \\ S^T & 0 \end{bmatrix}. \tag{2}$$

Let v^* be the optimal value of (1).

(b) Consider the semidefinite program:

$$\underset{X \in \mathbf{S}^{n+m}}{\text{maximise}} \quad \text{Tr}(AX) \quad \text{subject to} \quad X \succeq 0 \text{ and } X_{ii} = 1, \ \forall i = 1, \dots, n+m \tag{3}$$

where $\mathbf{S}^{n+m}$ is the space of real symmetric matrices of size $n+m$. Let p^*_{SDP} be the optimal value of (3). Show that $p^*_{SDP} \geqslant v^*$.

The goal of the remaining questions is to show that there is a constant $c_K \approx 0.5610$ such that $c_K \cdot p^*_{SDP} \leqslant v^*$. To do so we will use a randomised rounding scheme similar to the one we saw in lecture for the maximum cut problem.

(c) Show that if $a, b \in \mathbb{R}$ with $a \geqslant |b|$ then the matrix $\begin{bmatrix} aJ_{n,n} & bJ_{n,m} \\ bJ_{m,n} & aJ_{m,m} \end{bmatrix} \in \mathbf{S}^{n+m}$ is positive semidefinite where $J_{p,q}$ is the $p \times q$ matrix where all the entries are equal to one.

(d) Let $f, g : [-1,1] \to \mathbb{R}$ be two functions that admit a series expansion

$$f(x) = \sum_{k=0}^{\infty} f_k x^k \quad \text{and} \quad g(x) = \sum_{k=0}^{\infty} g_k x^k$$

and assume that $f_k \geqslant |g_k|$ for all $k \in \mathbb{N}$. Let also X be a block matrix $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \in \mathbf{S}^{n+m}$ where $X_{11} \in \mathbf{S}^n, X_{22} \in \mathbf{S}^m$ and $X_{12} \in \mathbb{R}^{n\times m}$, and assume that all the entries of X are in $[-1,1]$. Define

$$Y = \begin{bmatrix} f[X_{11}] & g[X_{12}] \\ g[X_{12}]^T & f[X_{22}] \end{bmatrix} \tag{4}$$

where $f[X_{11}]$ is the matrix obtained by applying the function f to each entry of X_{11}, and similarly for the other blocks. Show that if X is positive semidefinite then Y is positive semidefinite. To answer this question you can use the Schur product theorem without proof:

> *Schur product theorem*: If $P \succeq 0$ and $Q \succeq 0$ then $P \odot Q \succeq 0$ where $P \odot Q$ is the entrywise product of P and Q (i.e., $(P \odot Q)_{ij} = P_{ij} Q_{ij}$).

(e) Let X be an optimal solution of (3). Define Y as in (4) with the following choice of f and g:

$$f(x) = \sinh(c_K \pi x/2) \quad \text{and} \quad g(x) = \sin(c_K \pi x/2)$$

where

$$c_K = \frac{2}{\pi} \sinh^{-1}(1) = \frac{2}{\pi} \log(1 + \sqrt{2}) \approx 0.5610.$$

Verify that $Y \succeq 0$ and $Y_{ii} = 1$ for all $i = 1, \ldots, n+m$ [*Hint: the series expansion of* sinh *and* sin *are* $\sinh(x) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}$ *and* $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$].

(f) Let $v_1, \ldots, v_{n+m} \in \mathbb{R}^r$ with $r = \operatorname{rank}(Y)$ such that $Y_{ij} = \langle v_i, v_j \rangle$ for all $i, j = 1, \ldots, n+m$, and let y be the random variable in $\{-1, 1\}^{n+m}$ defined by:

$$y_i = \operatorname{sign}(\langle v_i, Z \rangle)$$

where Z is a standard Gaussian random vector in $\mathbb{R}^r$. We saw in class that

$$\mathbb{E}[y_i y_j] = \frac{2}{\pi} \arcsin(\langle v_i, v_j \rangle) \quad \forall 1 \leqslant i, j \leqslant n+m.$$

Show that

$$v^* \geqslant \mathbb{E}[y^T A y] = \frac{2}{\pi} \operatorname{Tr}(A \arcsin[Y])$$

where $\arcsin[Y]$ is the matrix obtained by applying the arcsin function to each entry of Y.

(g) Recalling the definition of A in (2), the definition of Y in (4) and the fact that $g(x) = \sin(c_K \pi x/2)$, show that $\operatorname{Tr}(A \arcsin[Y]) = c_K \frac{\pi}{2} \operatorname{Tr}(AX)$. Conclude that

$$v^* \geqslant c_K \operatorname{Tr}(AX) = c_K \cdot p^*_{SDP}.$$

3

Let $\mathbb{T}$ denote the unit circle in the complex plane

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

If $z = a+ib \in \mathbb{C}$ we denote by $\bar{z} = a - ib$ the complex conjugate of z and by $|z| = \sqrt{a^2+b^2}$ its modulus. A *trigonometric polynomial* of degree d is a function defined on $\mathbb{C} \setminus \{0\}$ of the form:

$$p(z) = \sum_{k=-d}^{d} p_k z^k$$

where $p_{-d}, \ldots, p_d \in \mathbb{C}$.

(a) Consider the trigonometric polynomial $p(z) = z^{-1} + 2 + z$. Show that $p(z) \geqslant 0$ for all $z \in \mathbb{T}$. Show that there exists a polynomial $q(z) = q_0 + q_1 z$ of degree 1 such that $p(z) = |q(z)|^2$ for all $z \in \mathbb{T}$. [*Hint: recall that if* $|z| = 1$ *then* $z^{-1} = \bar{z}$].

(b) In the next three questions we will show that any trigonometric polynomial p that is non-negative on $\mathbb{T}$ can be written as $p(z) = |q(z)|^2$ for $z \in \mathbb{T}$, for some polynomial q. Let thus $p(z)$ be a trigonometric polynomial that satisfies $p(z) \geqslant 0$ for all $z \in \mathbb{T}$. We also assume that $p_d \neq 0$.

(i) Prove that $p_{-k} = \overline{p_k}$ for all $k = 0, \ldots, d$ using the fact that $p(z) \in \mathbb{R}$ for all $z \in \mathbb{T}$. Use this to show that for any $z \in \mathbb{C} \setminus \{0\}$, $p(z^{-1}) = \overline{p(\overline{z})}$.

(ii) Let $P(z) = z^d p(z)$ and note that P is a complex polynomial of degree $2d$. Show that if $z \neq 0$ is a root of P then $1/\bar{z}$ is also a root of P.

(iii) Assuming without proof that any root in $\mathbb{T}$ of P has even multiplicity we can factorise P as follows:

$$P(z) = p_d \prod_{i=1}^{d} (z - z_i)(z - 1/\bar{z}_i) \quad \forall z \in \mathbb{C}. \tag{1}$$

Show using (1) that there exists a constant $c \in \mathbb{C}$ such that

$$p(z) = c \prod_{i=1}^{d} (z - z_i)(\bar{z} - \bar{z}_i) \quad \forall z \in \mathbb{T}.$$

Using the fact that p is not identically zero on $\mathbb{T}$ show that $c > 0$. Conclude that there exists a polynomial $q(z) = q_0 + q_1 z + \cdots + q_d z^d$ such that $p(z) = |q(z)|^2$ for all $z \in \mathbb{T}$.

(c) A matrix $M \in \mathbb{C}^{n \times n}$ is called *Hermitian* if $M_{ij} = \overline{M_{ji}}$ for all $1 \leqslant i, j \leqslant n$. A Hermitian matrix is called *positive semidefinite* if

$$\sum_{1 \leqslant i,j \leqslant n} M_{ij} \overline{x_i} x_j \geqslant 0 \quad \forall x \in \mathbb{C}^n.$$

(i) Show that if p is a trigonometric polynomial that satisfies $p(z) = |q(z)|^2$ for all $z \in \mathbb{T}$ for some polynomial $q(z) = \sum_{k=0}^{d} q_k z^k$, then there exists a Hermitian positive semidefinite matrix M of size $d+1$ such that

$$\sum_{\substack{0 \leqslant i,j \leqslant d \\ i-j=k}} M_{i,j} = p_k \quad \forall k = -d, \ldots, d \tag{2}$$

where the rows and columns of M are indexed by $0, \ldots, d$.

(ii) Conversely show that if there exists a Hermitian positive semidefinite matrix M that satisfies (2) then the trigonometric polynomial $p(z) = \sum_{k=-d}^{d} p_k z^k$ is non-negative on $\mathbb{T}$.

END OF PAPER