UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2017 1:30 pm to 3:30 pm

PAPER 337

CONVECTION AND MAGNETOCONVECTION

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The dimensionless equations for two-dimensional Boussinesq convection between stress-free, perfectly conducting boundaries for very small Prandtl number have the dimensionless form

$$0 = -\nabla P + R\theta \boldsymbol{e}_z + \nabla^2 \boldsymbol{u}; \quad \nabla \cdot \boldsymbol{u} = 0,$$
$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta = w + \nabla^2 \theta.$$

(i) Explain the meaning of the various symbols that appear, and give the boundary conditions to be satisfied by $\boldsymbol{u}$ and θ.

(ii) The boundary condition on θ at $z = 0$ is now replaced by

$$\theta(x, 0) = \gamma \epsilon^2 \cos 2k_c x, \quad \epsilon \ll 1,$$

where k_c is the critical wavenumber for the onset of convection when $\gamma = 0$. This non-uniform temperature drives a steady flow and temperature perturbation such that $w = w_0 = \epsilon^2 w_\epsilon(z) \cos 2k_c x$, $\theta = \theta_0 = \epsilon^2 \theta_\epsilon(z) \cos 2k_c x$. Write down at leading order in ϵ the equations and boundary conditions governing $w_\epsilon, \theta_\epsilon$ (solution of these equations is not required).

(iii) Now suppose that $R = R_c + \epsilon^2 R_2$, $\partial/\partial t = \epsilon^2 \partial/\partial \tau$, $w = w_0 + \epsilon w_1 + \ldots$, $\theta = \theta_0 + \epsilon \theta_1 + \ldots$, where $w_1 = A(\tau) f(z) e^{ik_c x} + \text{c.c.}$, $\theta_1 = A(\tau) g(z) e^{ik_c x} + \text{c.c.}$. By substituting into the equations, show that A will obey an equation of the form

$$\frac{dA}{d\tau} = \mu A + \beta A^* - \lambda |A|^2 A,$$

where $\mu \propto R_2$, $\beta \propto \gamma$. (It is not required to calculate the coefficients; but you should indicate how to find them).

Solve this equation to find any possible steady stable solutions, assuming that μ, β, λ are real, with $\lambda > 0$.

2

A feature of two-dimensional magnetoconvection in a layer with vertical magnetic field boundary conditions on the horizontal boundaries, and periodic lateral boundary conditions at $x = -\frac{L^*}{2}, \frac{L^*}{2}$ is the existence of a conserved quantity, the total magnetic flux $\int_{-\frac{L^*}{2}}^{\frac{L^*}{2}} \int_0^1 B_z(x,z)\, dxdz$. In this case the usual Landau-Ginzburg equation for modulated disturbances near the onset of steady convection can be described by two equations representing the evolution of the (complex) amplitude $A(X,T)$ of the convection modes and the long-wavelength modulation $B(X,T)$ of the vertical flux. These can be shown to take the form, after scaling,

$$\dot{A} = A + A_{XX} - A|A|^2 - AB,$$
$$\dot{B} = \sigma B_{XX} + \mu(|A|^2)_{XX},$$

where we assume $\int_{-\frac{L}{2}}^{\frac{L}{2}} B(X,T)dX = 0$, and L is the scaled spatial period, and where $\sigma > 0$.

(a) Consider the linear stability of uniform wavelike solutions of the form $A = Re^{\mathrm{i}qX}$, $R^2 = 1 - q^2$, $B = 0$, by writing $A = Re^{\mathrm{i}qX}(1 + a_1 e^{\mathrm{i}lX+\lambda T} + a_2^* e^{-\mathrm{i}lX+\lambda^* T})$, $B = be^{\mathrm{i}lX+\lambda T} + b^* e^{-\mathrm{i}lX+\lambda^* T}$. By finding a cubic equation for λ, show that when $|l| \ll 1$ there are small roots $\lambda = l^2\nu$, where $R^2\nu^2 - ((\mu-\sigma-1)R^2+2q^2)\nu - ((\mu-\sigma)R^2+2\sigma q^2) = 0$. Deduce that there is instability if

$$\frac{\mu}{\sigma} > \frac{1-3q^2}{1-q^2}.$$

(b) Now consider only real A, and define $\kappa = \mu/\sigma$. Show that for large cL there is an approximate steady solution $A = R\,\mathrm{sech}(cX)$, $B = \kappa(\langle A^2\rangle - A^2)$, where $\int_{-\frac{L}{2}}^{\frac{L}{2}} A^2\, dX = L\langle A^2\rangle$, and $\kappa\langle A^2\rangle = 1 + c^2$, $\kappa = 1 + 2c^2/R^2$. Show that for large cL, $cL\langle A^2\rangle \approx 2R^2$, and hence show that c satisfies the equation

$$L(c^2+1) = \frac{4c\kappa}{\kappa - 1}.$$

Deduce that such a solution can exist only if $1 < \kappa < L/(L-2)$.

3

Consider two-dimensional flow in a porous medium confined to a square domain of (dimensionless) side unity with $0 < x, z < 1$ with *all* sides perfect conductors. As is usual, there is a uniform vertical temperature gradient in the absence of motion. By writing the velocity as $\mathbf{u}(x, z) = \nabla \times \psi \hat{\mathbf{y}}$, justify the dimensionless equations

$$0 = -R\frac{\partial \theta}{\partial x} + \nabla^2 \psi, \quad \dot{\theta} = \frac{\partial \psi}{\partial x} + \nabla^2 \theta,$$

for the evolution of small disturbances, with $\psi = \theta = 0$ on all four boundaries where R is the Rayleigh number and θ the temperature perturbation.

(a) By writing $P = \psi + iq\theta$ where $q = \sqrt{R}$, or otherwise, show that the critical value of R for marginal stability is $R_c = q_c^2 = 8\pi^2$, but that there are two independent eigenfunctions, one with ψ even and θ odd about $x = 1/2$ and the other with θ even and ψ odd.

(b) Now suppose that $q = q_c + \epsilon q_1$, and $\dot{\theta} = \epsilon \lambda \theta$, $(\psi, \theta) = (\psi_0 + \epsilon \psi_1 + \ldots, \theta_0 + \epsilon \theta_1 + \ldots)$, where (ψ_0, θ_0) is one of the eigenfunctions found in (a). Verify the solvability condition (where $\langle \cdot \rangle \equiv \int_0^1 \int_0^1 \cdot \, dx\, dz$)

$$\langle \big(\psi_0 (-q_c^2 \theta_{1x} + \nabla^2 \psi_1) + q_c^2 \theta_0 (\psi_{1x} + \nabla^2 \theta_1) \big) \rangle = 0$$

and show that at leading order

$$\lambda = \frac{2q_1}{q_c} \frac{\langle \theta_0^2 \rangle}{\langle |\nabla \theta_0|^2 \rangle}$$

4

Write an essay on the onset of convection in a rotating layer. Your essay should cover the linear stability problem, the equation for the complex growth rate, wavenumber selection, interactions of oscillatory and steady convection and the instability of weakly nonlinear roll solutions.

END OF PAPER