UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 6 June, 2017 1:30 pm to 3:30 pm

PAPER 336

PERTURBATION METHODS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) For each real fixed $\alpha > 0$ and positive $\epsilon \to 0$, find two terms in the asymptotic expansions of the eigenvalues of the following matrix

$$\begin{pmatrix} -1 & 2 & 0 \\ \epsilon & \alpha & 1 \\ 0 & \epsilon & 1 \end{pmatrix}.$$

Identify a distinguished limit, and again find two terms in the asymptotic expansions of the eigenvalues. Show that these expansions match with your previous results.

(b) By using the method of multiple scales, find the leading-order approximation to the Mathieu equation

$$\ddot{x} + \left(1 + k\epsilon^2 + \epsilon \cos t\right) x = 0\,,$$

for real $k = O(1)$ and positive $\epsilon \to 0$, valid up to long times $t = O(\epsilon^{-\beta})$, where $\beta > 0$ is to be determined.

For what range of values of k is the solution always stable for any initial conditions?

2

(a) The integral $I(\gamma;\varepsilon)$ is defined for real $\gamma > 0$ by

$$I(\gamma;\varepsilon) = \int_0^1 \frac{1}{(\gamma(x-1)+1)^2 + (\varepsilon \log(2-x))^2}\, dx \, .$$

If $0 < \varepsilon \ll 1$ calculate the leading-order asymptotic approximations of $I(\gamma;\varepsilon)$ for

(i) $0 < \gamma < 1$,

(ii) γ close to 1 (where how close to 1 should be specified),

(iii) $\gamma > 1$.

Briefly discuss whether the approximation for $\gamma > 1$ is uniformly valid for $\gamma \gg 1$.

(b) Let

$$J(\lambda) = \int_{\mathcal{C}} \frac{F(z)}{z - z_0} \exp(-i\lambda G(z))\, dz \ ,$$

where F and G are analytic near the contour $\mathcal{C}$ and z_0, $G(z)$ has a single saddle point at $z = z_0$, and $\mathcal{C}$ passes from one 'valley' of $\Im(G)$ with respect to z_0 to another, avoiding z_0 in a clockwise manner. Show that for $\lambda \gg 1$,

$$J(\lambda) \sim -i\pi F(z_0) \exp(-i\lambda G(z_0)) \ .$$

What is the order of the next correction?

3

For $0 \leqslant x \leqslant 1$, the function $y(x;\epsilon)$ satisfies the differential equation

$$\epsilon x^{\frac{1}{2}} \sin(x+\epsilon)\, y_{xx} + (x+\epsilon)(x+1)\, y_x - (x+\epsilon^2)y = 0 \ ,$$

and the boundary conditions

$$y(0;\epsilon) = 0 \ , \quad y(1;\epsilon) = 2 \ .$$

On the assumption that $0 < \epsilon \ll 1$, find the solution correct to, and including, terms of $O(\epsilon)$ in three asymptotic regions (which are to be identified). The following integral may prove useful

$$2\int_0^x \exp(-2q^{\frac{1}{2}})\, dq = 1 - (1 + 2x^{\frac{1}{2}}) \exp(-2x^{\frac{1}{2}}) \ .$$

END OF PAPER