UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 12 June, 2017 9:00 am to 12:00 pm

PAPER 332

FLUID DYNAMICS OF THE SOLID EARTH

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Carbon dioxide (CO_2), with density ρ_c and viscosity μ_c is injected into the top of an aquifer with porosity ϕ and permeability k which is initially saturated in water of density $\rho_w > \rho_c$ and viscosity $\mu_w > \mu_c$. After an initial transient, the buoyant CO_2 forms a uniform, horizontal interface with the water. Continued injection then causes the interface to propagates downwards with velocity $(u, w) = (0, -W)$ where W is a positive constant. Analyse the stability of the interface and determine the growth rate of instabilities as a function of their wavenumber α.

Determine the critical speed W_c below which the interface is stable to perturbations and above which the interface is unstable.

In a reservoir for which the permeability $k \simeq 10^{-12}$ m^2, and for which $\rho_c \simeq$ 700 kg m^{-3}, $\rho_w \simeq 1200$ kg m^{-3}, $\mu_c \simeq 5 \times 10^{-4}$ Pa s and $\mu_w \simeq 2 \times 10^{-3}$ Pa s find the critical velocity W_c (in m/yr).

2

At time $t = 0$, a large block of ice of uniform temperature $T_{-\infty}$ is brought into contact with a large body of salt solution of uniform concentration C_0 and temperature $T_\infty > -mC_0$, where the liquidus temperature $T_L(C) = -mC$ of the salt water is assumed to be linear. Initially there is a planar interface between between the ice and solution.

Determine or simply write down a similarity solution for the subsequent evolution of the planar interface such that its displacement into the liquid $a(t) = 2\lambda\sqrt{Dt}$, where D is the diffusivity of salt in solution, deriving particularly the system of algebraic equations that determine λ and the interfacial temperature T_i.

Approximate the equations for the case that $\lambda = \mathrm{O}(1)$ when $\epsilon^2 \equiv D/\kappa \ll 1$, where κ is the thermal diffusivity. Define the undercooling $\theta_- \equiv -mC_0 - T_{-\infty}$ and the superheat $\theta_+ \equiv T_\infty + mC_0$, and show that, to leading order in ϵ, the interfacial temperature

$$T_i = \frac{T_\infty + T_{-\infty}}{2}$$

and that

(i) the salty water solidifies if $\theta_- > \theta_+$;

(ii) there is constitutional supercooling in the salty water if $\theta_- > \theta_+(1 + 2\epsilon)$.

3

A lava lake is filled to a depth H with a lava enriched with a very long-lived radioactive element, and the radioactive decay thereby provides an approximately constant internal heating Q (W/kg). The surface is radiatively cooled to a fixed temperature less than the melting temperature of the lava, $T_s < T_m$, and thus drives the formation of a crust of thickness $a(t) \ll H$.

Vigorous convection ensures that the lava beneath the crust is well mixed, with average temperature $\overline{T}(t)$. Assume that when convection is vigorous and driven primarily by the radiogenic heating the heat flux from the lava to the crust is approximately

$$F = \lambda k \left(\frac{\rho_0 g \alpha Q}{k \kappa \mu} \right)^{1/5} (\overline{T} - T_m),$$

where λ is a constant, k is the thermal conductivity, κ the thermal diffusivity, α is the thermal coefficient of expansion, ρ_0 a reference density, μ the viscosity of the lava, and g the acceleration due to gravity.

Derive expressions for the evolution of the crustal thickness $a(t)$ and average temperature of the lava $\overline{T}(t)$, including the effects of latent heat release on solidification, (linear) heat conduction through the crust, convective heat losses from the chamber, and radiogenic heat production. You may assume radiogenic heating is negligible within the crust.

Using the thermal diffusion timescale, scale the problem and identify the three key non-dimensional parameters governing the evolution of the lava lake. The evolution of the chamber depends critically on the initial temperature of the lava. Find the initial condition for which the lava temperature remains constant for all time and in this limit derive an implicit analytical expression for the crustal thickness as a function of time.

For temperatures either greater than or less than this temperature determine the full evolution of the crustal thickness and temperature at early and intermedia times, taking care to plot and interpret your results.

4

A two-dimensional marine ice sheet of density ρ, dynamic viscosity μ and overall thickness $H(x,t)$ flows over bedrock that is a depth $b(x) = \alpha x$ below sea level and has surface elevation $h(x,t)$ above sea level, where x is horizontal displacement and t is time. It flows into an ocean of density ρ_w across a grounding line at $x = x_G(t)$ to form an ice shelf, which you may assume is un-buttressed. A negligibly-thin layer of till (a mixture of clay and pebbles that can be considered here to be a Newtonian fluid) of dynamic viscosity $\lambda\mu$ ($\lambda \ll 1$) and thickness l lubricates the base of the ice sheet. The ice therefore has negligible internal shear.

Use force and mass balances to derive the governing equations

$$4\mu \frac{\partial}{\partial x}\left(H\frac{\partial u}{\partial x}\right) - \rho g H \frac{\partial h}{\partial x} - \lambda\mu\frac{u}{l} = 0,$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(Hu) = 0,$$

where $u(x,t)$ is the vertically-uniform flow in the ice sheet and g is the acceleration due to gravity.

What two boundary conditions must be applied at the grounding line $x = x_G(t)$?

How may the equations be simplified in the limit

$$\frac{l}{H}\frac{H^2}{L^2} \ll \lambda,$$

where $L \gg H$ is a characteristic length scale for variations in the horizontal direction?

Using this simplification from now on, determine the evolution equation for the grounding line

$$\left[\alpha\frac{\rho_w}{\rho} - \frac{\partial H}{\partial x}\right]\dot{x}_G = \frac{l}{\lambda}\frac{g}{\nu}H\frac{\partial h}{\partial x}\left(\frac{\partial h}{\partial x} + \alpha\right) - \frac{1}{8}\frac{g'}{\nu}H^2 \quad \text{at} \quad x = x_G(t),$$

where $g' = (\rho_w - \rho)g/\rho_w$ is a reduced gravity.

Show that in a steady state with the ice sheet having volume flux q_0 per unit width,

$$\frac{g'}{8g}\mathcal{H}^5 = \mathcal{Q}\left(\mathcal{Q} - \alpha\mathcal{H}^2\right),$$

where

$$\mathcal{H} = \frac{\lambda}{l}H, \qquad \mathcal{Q} = \frac{\lambda^3}{l^3}\frac{\nu}{g}q_0.$$

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