UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 2 June, 2017 1:30 pm to 4:30 pm

PAPER 331

HYDRODYNAMIC STABILITY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider an isolated interface (with no surface tension) between semi-infinite layers of two inviscid fluids of different densities $\rho_1 < \rho_2$, moving in the $x-$direction at two (different) constant speeds U_1 and U_2. The displacement $\eta = B\exp[ik(x-ct)]$ (real part understood, with $k > 0$ real) of the interface away from its equilibrium position at $z = 0$ may be assumed to be sufficiently small and smooth that the problem may be linearised, and the problem can also be considered to be two-dimensional.

(i) Write down the appropriate conditions on the upper-layer velocity potential ϕ_1 as $z \to \infty$ and on the lower-layer velocity potential ϕ_2 as $z \to -\infty$.

(ii) Briefly explain why the appropriate boundary conditions to apply at $z = \eta$ are

$$\begin{aligned} \left.\frac{\partial \phi_{1,2}}{\partial z}\right|_{z=\eta} &= \frac{D\eta}{Dt}, \\ \rho_1 \frac{\partial \phi_1}{\partial t} + \frac{1}{2}\rho_1 \left|\boldsymbol{\nabla}\phi_1\right|^2 + g\rho_1\eta &= \rho_2 \frac{\partial \phi_2}{\partial t} + \frac{1}{2}\rho_2 \left|\boldsymbol{\nabla}\phi_2\right|^2 + g\rho_2\eta. \end{aligned}$$

(iii) Linearise these boundary conditions, and hence show that there is instability for

$$k > \frac{g(\rho_2^2 - \rho_1^2)}{\rho_1\rho_2(U_1 - U_2)^2}.$$

(b) Now consider an infinite constant density fluid, with two-dimensional background velocity distribution in the form of a top-hat jet in the $x-$direction with $U = V$ for $|z| < L$ and $U = 0$ for $|z| > L$. Assume that the perturbation velocity potential $\phi' \propto \exp[ik(x-ct)]$, and that once again perturbations are sufficiently small and smooth so that the problem may be linearised.

(i) Determine the appropriate conditions on ϕ' at $z = \pm L$.

(ii) Assuming that ϕ' is an odd function of z, obtain the dispersion relation:

$$c^2 = -(V-c)^2 \tanh kL,$$

and hence deduce that the flow is unstable for all choices of k.

(iii) Obtain the equivalent dispersion relation if ϕ' is assumed to be an even function of z.

(iv) At a fixed wavenumber, compare the growth rates associated with the odd and even velocity potentials.

(v) At fixed wavenumber, show that the growth rates of perturbations for the top-hat jet are never larger than the growth rates for perturbations at a single interface between two fluids of the same density, one of which is stationary and the other of which is moving at velocity V.

2

Consider infinitesimal two-dimensional perturbations about a parallel shear flow in an inviscid stratified fluid:

$$\begin{aligned}
\mathbf{u} &= \overline{U}(z)\hat{\mathbf{x}} + \mathbf{u}'(x,z,t),\\
p &= \overline{p}(z) + p'(x,z,t),\\
\rho &= \overline{\rho}(z) + \rho'(x,z,t),\\
[\mathbf{u}', p', \rho'] &= [\hat{\mathbf{u}}(z), \hat{p}(z), \hat{\rho}(z)]\exp[ik(x-ct)],
\end{aligned}$$

where the wavenumber k is assumed real, and the phase speed c may in general be complex. Upon appropriate application of the Boussinesq approximation, the vertical velocity eigenfunction $\hat{w}$ satisfies the Taylor-Goldstein equation,

$$\left(\frac{d^2}{dz^2} - k^2\right)\hat{w} - \frac{\hat{w}}{(\overline{U}-c)}\frac{d^2}{dz^2}\overline{U} + \frac{N^2\hat{w}}{(\overline{U}-c)^2} = 0;\ N^2 = -\frac{g}{\rho_0}\frac{d\overline{\rho}}{dz},$$

where N is the buoyancy frequency and ρ_0 is an appropriate reference density.

(a) Show that a necessary condition for instability (i.e. for $c_i > 0$) is that

$$N^2 - \frac{1}{4}\left(\frac{d\overline{U}}{dz}\right)^2 < 0,$$

somewhere in a flow, with $\hat{w}/[\overline{U}-c]^{1/2}$ zero on the $z-$boundaries of the flow domain.

(b) Now consider a flow where $N^2 = J\,\mathrm{sech}^2 z$ and $\overline{U} = \tanh z$ for $-\infty < z < \infty$. Assume that

$$\hat{w}_k(z) = (\mathrm{sech} z)^k(\tanh z)^{1-k},$$

for $0 \leqslant k \leqslant 1$.

(i) Derive a condition on J such that $\hat{w}_k(z)$ is a solution of the Taylor-Goldstein equation corresponding to a neutral perturbation with $c = 0$.

(ii) Briefly discuss this result in terms of the 'Miles-Howard' theorem proved in part (a).

3

(a) First consider the linear complex Ginzburg-Landau equation:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\psi_L - \mu\psi_L - (1 + ic_d)\frac{\partial^2}{\partial x^2}\psi_L = 0.$$

Assume that ψ_L describes an infinitesimal wave-like perturbation, and so is proportional to $\exp[i(kx - \omega t)]$, where k and ω are in general complex.

(i) Express the dispersion relation in the form

$$\omega = \omega_0 + (c_d - i)(k - k_0)^2,$$

where the absolute frequency ω_0 and absolute wavenumber k_0 are to be determined.

(ii) By considering the group velocity when $k = k_0$, identify criteria on the absolute frequency for the flow to be convectively or absolutely unstable.

(b) Now consider the nonlinear Ginzburg-Landau equation with $\mu > k^2$ and no dispersion:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\psi = \mu\psi + \frac{\partial^2}{\partial x^2}\psi - |\psi|^2\psi.$$

(i) Show that $\psi_S = Q\exp[ik(x - Ut)]$ is a solution of the nonlinear Ginzburg-Landau equation, where $Q > 0$ is a constant to be determined.

(ii) Now determine the non-trivial time-dependent function $R(t)$ with initial value $R(0) = R_0 > 0, R_0 \neq Q$, such that $\psi_R = R(t)\exp[ik(x - Ut)]$ is a solution of the nonlinear Ginzburg-Landau equation.

(iii) Hence show for all choices of $R_0 \neq Q$ that $\psi_R \to \psi_S$ monotonically as $t \to \infty$.

(iv) If $R \ll Q$, show that ψ_R has approximately the same exponential growth rate as the solutions ψ_L of the linear complex Ginzburg-Landau equation considered in (a), with $c_d = 0$ and $\mu > k^2$.

4

Consider plane Poiseuille flow of a zero-mean dynamic scalar field $\theta(\mathbf{x},t)$ in a channel $-1 \leqslant y \leqslant 1$ with base steady flow $\overline{\mathbf{u}} = (1-y^2)\hat{\mathbf{x}}$ and base scalar distribution $\overline{\theta} = -\mathrm{erf}(30y)$. For finite Péclet number Pe, Reynolds number Re and bulk Richardson number Ri_B, the evolution equations are:

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{U}\cdot\boldsymbol{\nabla}\mathbf{u} + \mathbf{u}\cdot\boldsymbol{\nabla}\overline{\mathbf{u}} + Ri_B\theta\hat{\mathbf{y}} + \boldsymbol{\nabla}p - Re^{-1}\boldsymbol{\nabla}^2\mathbf{u} &= \mathbf{0}, \\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0, \quad \partial_t\theta + \mathbf{U}\cdot\boldsymbol{\nabla}\Theta - Pe^{-1}\nabla^2\Theta &= 0, \end{aligned}$$

where here the total velocity field is $\mathbf{U} = \overline{\mathbf{u}} + \mathbf{u}$, and the total scalar field is $\Theta = \overline{\theta} + \theta$. Impose

$$\mathbf{u}(x,\pm 1,z,t) = \mathbf{0};\ \partial_y p(x,\pm 1,z,t) = \partial_y\Theta(x,\pm 1,z,t) = \theta(\mathbf{x},0) = 0,$$

and periodicity at $\pm L_x$ and $\pm L_z$. Consider the augmented Lagrangian:

$$\begin{aligned} \mathcal{L} \;=\; & \Big(\Theta(\mathbf{x},T),\Theta(\mathbf{x},T)\Big) - \Big\langle \mathbf{u}^\dagger, \partial_t\mathbf{u} + \mathbf{U}\cdot\boldsymbol{\nabla}\mathbf{u} + \mathbf{u}\cdot\boldsymbol{\nabla}\overline{\mathbf{u}} + Ri_B\theta\hat{\mathbf{y}} + \boldsymbol{\nabla}p - Re^{-1}\nabla^2\mathbf{u}\Big\rangle \\ & -\Big\langle p^\dagger, \boldsymbol{\nabla}\cdot\mathbf{u}\Big\rangle - \Big\langle \theta^\dagger, \partial_t\theta + \mathbf{U}\cdot\boldsymbol{\nabla}\Theta - Pe^{-1}\nabla^2\Theta\Big\rangle - \Big(\mathbf{u}_0^\dagger, \mathbf{u}(\mathbf{x},0) - \mathbf{u}_0\Big), \end{aligned}$$

where $\mathbf{u}^\dagger$, $p^\dagger$, $\theta^\dagger$ and $\mathbf{u}_0^\dagger$ are appropriate Lagrange multipliers, $\mathbf{u}_0$ is a specified velocity initial condition, T is the target time, and the scalar products $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ are defined as

$$\Big\langle \mathbf{a}(\mathbf{x},t),\mathbf{b}(\mathbf{x},t)\Big\rangle = \int_0^T \Big(\mathbf{a}(\mathbf{x},t),\mathbf{b}(\mathbf{x},t)\Big)\mathrm{d}t = \int_0^T\int_{-L_z}^{L_z}\int_{-1}^{1}\int_{-L_x}^{L_x} \mathbf{a}(\mathbf{x},t)\cdot\mathbf{b}(\mathbf{x},t)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t.$$

(a) Derive the adjoint equations to identify an optimal initial perturbation to the velocity field with a fixed kinetic energy E_0 to minimise scalar variance at terminal time $t = T$.

(b) Identify the appropriate initial $t = 0$ and terminal $t = T$ conditions for the adjoint variables for such a locally optimal initial velocity perturbation.

END OF PAPER