UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 2 June, 2017 9:00 am to 12:00 pm

PAPER 329

SLOW VISCOUS FLOW

*Attempt no more than **THREE** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) A force-free couple-free spherical micro-organism of radius a can swim through fluid of the same density by using a surface layer of tiny flagella to prescribe a tangential relative velocity $\mathbf{u}_s(\mathbf{x})$ between the fluid just outside the organism and the rigid body of the organism. Hence if the velocity of the organism is $\mathbf{U}+\boldsymbol{\Omega}\wedge\mathbf{x}$, with $\mathbf{U}$ and $\boldsymbol{\Omega}$ constants, then the fluid velocity immediately outside the organism is $\mathbf{U}+\boldsymbol{\Omega}\wedge\mathbf{x}+\mathbf{u}_s(\mathbf{x})$.

The organism is swimming through unbounded fluid, which is otherwise at rest, and prescribing $\mathbf{u}_s(\mathbf{x}) = (\mathbf{I}-\mathbf{nn})\boldsymbol{\cdot}(\mathbf{A}+\mathbf{B}\boldsymbol{\cdot}\mathbf{n})$, where $\mathbf{A}$ is a constant vector, $\mathbf{B}$ is a constant symmetric traceless second-rank tensor and $\mathbf{n}(\mathbf{x}) = \mathbf{x}/a$ is the outward normal to the organism. Explain, without calculation, why $\mathbf{U}$ is just a multiple of $\mathbf{A}$. What is the value of $\boldsymbol{\Omega}$, and why?

State the Papkovich–Neuber representation for the velocity and pressure in Stokes flow. Use this representation, explaining your choice of trial harmonic potentials, to determine the velocity field $\mathbf{u}(\mathbf{x})$ outside the organism and hence obtain the value of the swimming velocity $\mathbf{U}$.

(b) The organism is now swimming with the same $\mathbf{u}_s$ in the presence of a rigid sphere of radius a that is held stationary at $\mathbf{x}=\mathbf{X}$. Assuming that $a \ll R$, where $R = |\mathbf{X}|$, find the *leading-order* approximations to the force and the couple that must be applied to the rigid sphere to keep it stationary. State the order of the next correction to the force, and state where it comes from.

Find the leading-order correction to the swimming velocity of the organism due to the presence of the stationary sphere, and state the order of the next correction.

Write down the velocity field due to a Stokeslet of strength $\mathbf{F}$, and show that the corresponding vorticity is $\mathbf{F}\wedge\mathbf{x}/4\pi\mu r^3$. Deduce that the organism does not rotate with an $O(a^3B/R^4)$ angular velocity, but with a smaller angular velocity, given at leading order by

$$\boldsymbol{\Omega} = \frac{\mathbf{A}\wedge\mathbf{X}a^4}{4R^6}\,.$$

[*You may assume the Faxén formulae*

$$\mathbf{U} = \frac{\mathbf{F}}{6\pi\mu a} + \mathbf{u}_\infty + \frac{a^2}{6}\nabla^2\mathbf{u}_\infty\,, \quad \boldsymbol{\Omega} = \frac{\mathbf{G}}{8\pi\mu a^3} + \frac{1}{2}\boldsymbol{\omega}_\infty\,,$$

but should explain how you apply them.]

2

(a) An axisymmetric thread of viscous fluid falls steadily and vertically from a nozzle through air of negligible viscosity and constant pressure p_e. The thread has radius $a(z)$, where $|\partial a/\partial z| \ll 1$. Surface tension acts on the interface, but inertia is negligible. Derive the equations governing a and the vertical velocity w, explaining your argument carefully. [Hint: interfacial tension γ per unit length should be included in the vertical force balance.]

For the case of steady flow with constant flux $\pi a_0^2 w_0$ from a nozzle of radius a_0, show that

$$3\frac{d}{dZ}\left(\frac{1}{W}\frac{dW}{dZ}\right) + \frac{1}{W} + \Gamma\frac{dW^{-1/2}}{dZ} = 0\,, \qquad W(0) = 1\,,$$

where the dimensionless variables W, Z and Γ should be defined. Verify that there is a solution of the form $W = (kZ+1)^{\alpha}$, where α and $k(\Gamma)$ are to be found. Does surface tension increase or decrease the rate of fall? Explain this effect physically.

(b) A long annular cylinder of viscous fluid, $R_i(t) \leqslant r \leqslant R_e(t)$, undergoes uniform axial extension with velocity $w(z) = Ez$, where E is a constant. Surface tension acts on both interfaces, gravity and inertia are negligible, and the pressures, p_i and p_e, internal and external to the annulus are held constant. (The air inside the annulus is assumed to escape along the axis with a negligible pressure gradient.)

Determine the radial velocity $u(r,t)$ and show that the pressure $p = p(t)$ is given by

$$(-p - \mu E)(R_e^2 - R_i^2) = -R_e^2 p_e + R_i^2 p_i - \gamma(R_e + R_i)\,.$$

Use the kinematic boundary conditions to show that

$$\frac{d}{dt}(R_e - R_i) = -\frac{E}{2}(R_e - R_i) + \frac{\gamma}{2\mu} - \frac{(p_i - p_e)R_i R_e}{2\mu(R_i + R_e)}\,,$$

and to find an evolution equation for $A = R_e^2 - R_i^2$.

For the case $p_i = p_e$, $E = 0$ and $R_e(0) = 2R_i(0)$, find the time when $R_i(t) = 0$.

(c) A hollow axisymmetric thread of viscous fluid, $R_i(z) \leqslant r \leqslant R_e(z)$, falls steadily and vertically under gravity from an annular nozzle. The internal and external pressures are held constant. Adapt your previous analyses where necessary to show that

$$3\mu\frac{\partial}{\partial z}\left(A\frac{\partial w}{\partial z}\right) + \rho g A + \gamma\frac{\partial(R_e + R_i)}{\partial z} = 0\,.$$

For the case $p_i = p_e$, use the kinematic equations to show further that the central hole closes at the height z^* where

$$\frac{\gamma}{2\mu w_0^{1/2}}\int_0^{z^*}\frac{d\zeta}{w(\zeta)^{1/2}} = A(0)^{1/2} - R_e(0) + R_i(0)\,.$$

Without attempting a detailed calculation, use part (a) to explain briefly why you would expect z^* to be finite.

3

A thin film of viscous fluid flows with a typical velocity U between two rigid surfaces with a typical gap width H that varies on a lengthscale $L \gg H$. Use scaling arguments to show that the typical shear stress τ and the typical pressure variations P satisfy $\tau/P \ll 1$.

The annular gap between two nearly concentric, horizontal, rigid cylinders of radius a and $a+\Delta$, with $0 < \Delta \ll a$, is filled with fluid of viscosity μ. The cylinders are offset vertically, with axes at $(x,z) = (0,0)$ and $(0, \alpha\Delta)$ respectively, where $-1 < \alpha < 1$. Show that the gap thickness $h(\theta)$ between the cylinders is approximately given by

$$h = (1 + \alpha \sin\theta)\Delta,$$

where θ is the angle to the x-axis. [The effects of inertia and gravity are negligible.]

(a) The inner cylinder rotates with angular velocity Ω about its own axis, and the outer cylinder is stationary. Use lubrication theory to determine the flux q (per unit axial length) in the annular gap.

Express the local pressure gradient, and the local shear stresses on the inner and outer cylinders, in terms of q, Ω, a, h and μ.

Calculate the leading-order contribution to the horizontal force F_x (per unit length) required to hold the axis of the inner cylinder stationary, showing that

$$F_x = -\frac{6\pi\mu\Omega a^3}{\Delta^2}\frac{\alpha}{(1-\alpha^2)^{1/2}(1+\frac{1}{2}\alpha^2)}\,.$$

[*Hint:* Integrate by parts to avoid finding $p(\theta)$.] Explain, without calculation, why the vertical force required is zero.

Calculate the couple on each cylinder about its own axis. Why do these couples not add to zero? How does the difference relate to F_x?

(b) Now the outer cylinder moves upwards with speed $V = (d\alpha/dt)\Delta$, and the inner cylinder is stationary. Calculate the vertical force F_z (per unit length) required to hold the inner cylinder stationary.

(c) Model a lubricated axle bearing by modifying your calculations as follows. The inner cylinder is now able to move both horizontally and vertically, and rotates with angular velocity Ω. The outer cylinder is stationary. A external vertical load $-F$ (per unit length) is applied to the inner cylinder, but there is no applied horizontal load.

If the cylinders are initially concentric, what is the initial velocity of the inner cylinder? Briefly describe what happens next. Describe the equilibrium position of the inner cylinder?

[*You may assume that if* $I_n = \displaystyle\int_0^{2\pi} \frac{d\theta}{(1+A\sin\theta)^n}$ *then* $I_1 = \dfrac{2\pi}{(1-A^2)^{1/2}}$,

$$I_2 = \frac{2\pi}{(1-A^2)^{3/2}}, \quad I_3 = \frac{2\pi(1+\frac{1}{2}A^2)}{(1-A^2)^{5/2}}. \quad \textit{Also} \int_0^{2\pi} \frac{(1-\sin^2\theta)\,d\theta}{(1+A\sin\theta)^3} = \frac{\pi}{(1-A^2)^{3/2}}.$$]

END OF PAPER