UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 9 June, 2017 1:30 pm to 3:30 pm

PAPER 327

DISTRIBUTION THEORY AND APPLICATIONS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State and prove the Paley-Wiener-Schwartz theorem.

Let P be a non-zero polynomial of one variable with no repeated roots and define $\mathcal{N} = \{\varphi \in \mathcal{E}(\mathbf{R}) : P(-D)\varphi = 0\}$. Prove that the equation

$$P(D)u = v \in \mathcal{E}'(\mathbf{R})$$

has a solution $u \in \mathcal{E}'(\mathbf{R})$ if and only if $\langle v, \varphi \rangle = 0$ for all $\varphi \in \mathcal{N}$.

2

Define the space of Schwartz functions $\mathcal{S}(\mathbf{R}^n)$ and the space of tempered distributions $\mathcal{S}'(\mathbf{R}^n)$, specifying the notion of convergence in each.

Show that the Fourier transform defines a continuous isomorphism on $\mathcal{S}(\mathbf{R}^n)$, and that this extends to a continuous isomorphism on $\mathcal{S}'(\mathbf{R}^n)$.

For the remainder of this question assume $n > 1$. Let $R \in \mathrm{SO}(n)$ be a rotation matrix. For $T \in \mathcal{S}'(\mathbf{R}^n)$ define $T \circ R \in \mathcal{S}'(\mathbf{R}^n)$ by

$$\langle T \circ R, \varphi \rangle = \left\langle T, \varphi \circ R^t \right\rangle, \quad \varphi \in \mathcal{S}(\mathbf{R}^n).$$

T is said to be *radial* if $T \circ R = T$ for all $R \in \mathrm{SO}(n)$. Show that:

(i) $T \in \mathcal{S}'(\mathbf{R}^n)$ is radial if and only if $\hat{T}$ is radial.

(ii) If $\psi \in \mathcal{S}(\mathbf{R}^n)$ and $T \in \mathcal{S}'(\mathbf{R}^n)$ are both radial, so are ψT and $T * \psi$.

Hence, or otherwise, show that a radial tempered distribution can be written as the limit of a sequence of radial Schwartz functions.

3

State and prove the Malgrange-Ehrenpries theorem for $P(D)$, where P is a non-zero polynomial in n variables. Your proof should involve the construction of a suitable "Hörmander staircase".

Show that $P(D)u = f$ admits a smooth solution for every $f \in \mathcal{D}(\mathbf{R}^n)$. Deduce that all solutions to $P(D)u = f \in \mathcal{D}(\mathbf{R}^n)$ are smooth if and only if all solutions to $P(D)v = 0$ are smooth.

END OF PAPER