UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 5 June, 2017 1:30 pm to 4:30 pm

PAPER 326

INVERSE PROBLEMS IN IMAGING

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Linear inverse problems

This question deals with linear inverse problems, generalised solutions, the Moore–Penrose inverse and the singular value decomposition of compact operators.

1. (a) Write down the definitions of a *forward problem* and the associated *inverse problem*. When is an inverse problem *ill-posed*?

   (b) Write down the definition of *least squares solution* and *minimal norm solution*. Under what conditions do least squares solutions exist? Give an example where least squares solutions do not exist.

2. (a) Write down the definition of the *Moore–Penrose inverse* and state its connection to least squares solutions and the minimal norm solution. State an equivalent condition to the continuity of the Moore–Penrose inverse.

   (b) Consider the linear operator $K : \ell^2 \to \ell^2$, defined by

   $$(Ku)_j := u_j/j\,.$$

   Show that K is continuous and calculate its range, null space and Moore–Penrose inverse $K^\dagger$. It is necessary to also state the domain of $K^\dagger$. Is $K^\dagger$ continuous?

3. Let $\mathcal{U}, \mathcal{V}$ be Hilbert spaces and consider a linear and compact operator $K \in \mathcal{K}(\mathcal{U}, \mathcal{V})$.

   (a) Write down the definition of the *singular value decomposition (SVD)* of K and the SVD of the operator in 2b.

   (b) What is an equivalent condition to $f \in \mathcal{R}(K)$? Use this condition to verify for which $p > 0$ the data f with $f_j = j^{-p}$ is in the range of K as defined in 2b.

   (c) Consider now the special case $\mathcal{U} = L^2([0,1]), \mathcal{V} = L^2([0,1])$ with the integral operator $K : L^2([0,1]) \to L^2([0,1])$ defined as

   $$(Ku)(y) := \int_0^y u(x)\,dx\,.$$

   Let f be given by $f(x) := \begin{cases} 0 & x < \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases}$. Show that $f \in \overline{\mathcal{R}(K)} \setminus \mathcal{R}(K)$. Is the Moore–Penrose inverse of K continuous? *Hint: You can use without proof that the SVD of K is given by $\{u_j, v_j, \sigma_j\}_{j\in\mathbb{N}}$ with*

   $$u_j(x) = \sqrt{2}\cos(\sigma_j^{-1}x), \quad v_j(x) = \sqrt{2}\sin(\sigma_j^{-1}x), \quad \textit{and} \quad \sigma_j = \frac{2}{(2j-1)\pi}\,.$$

2 Regularisation

This question deals with various aspects of the concept of regularisation.

1. Write down the definitions of *regularisation*, *linear regularisation*, *parameter choice rule* and *convergent regularisation*. Give an example for a regularisation.

2. State the decomposition of the total error of the regularised solution with the minimal norm solution. Sketch the qualitative behaviour of the errors.

3. Consider the problem of differentiation with the integral operator $K : L^2([0,1]) \to L^2([0,1])$ defined as $(Ku)(y) := \int_0^y u(x)\,dx$. Approximate $K^\dagger$ with the one-sided differential quotient operator $D_h : L^2([0,1]) \to L^2([0,1])$ with

$$(D_h f)(x) := \frac{1}{h}\begin{cases}[f(x+h) - f(x)] & x \in [0, \frac{1}{2}) \\ [f(x) - f(x-h)] & x \in [\frac{1}{2}, 1]\end{cases},$$

for $h \in (0, 1/2)$. We consider exact data $f \in C^2([0,1])$ and noisy measurements $f^\delta \in L^2([0,1])$ for which $\|f - f^\delta\|_{L^2} \leqslant \delta$ holds true.

   (a) Verify the following estimate for the error between $K^\dagger f = f'$ and $D_h f^\delta$:

$$\|K^\dagger f - D_h f^\delta\|_{L^2} \leqslant \frac{\sqrt{6}\delta}{h} + \frac{\|f''\|_\infty}{2} h \tag{1}$$

   (b) Determine $h(\delta)$ that minimises the right-hand-side of (1). Find a parameter choice rule such that D_h is a convergent regularisation.

4. Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be Hilbert spaces, $K \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ be an injective, linear and bounded operator and $B \in \mathcal{L}(\mathcal{U}, \mathcal{W})$ be a linear and bounded operator with $\|Bu\| \geqslant \beta\|u\|$. Furthermore, let $f \in \mathcal{D}(K^\dagger)$ and $f^\delta \in \mathcal{V}$ with $\|f - f^\delta\| \leqslant \delta$. Then we define *Tikhonov–Philipps regularisation* as

$$R_\alpha f^\delta = (K^*K + \alpha B^*B)^{-1} K^* f^\delta \,.$$

   (a) Let $r, \alpha > 0$. Verify the following estimate

$$\|R_\alpha f - K^\dagger f\| \leqslant \eta_r + \alpha^{1/2}\beta r, \quad \eta_r := \inf\left\{\|\beta^{-2}B^*BK^\dagger f - K^*w\| \,\middle|\, w \in \mathcal{V}, \|w\| \leqslant r\right\}$$

   and show that $\lim_{r\to\infty} \eta_r = 0$. *Hint: Begin by estimating* $\|KR_\alpha f - KK^\dagger f\|^2$.

   (b) Use the result of 4a to show that Tikhonov–Philipps regularisation is a convergent regularisation with an appropriate parameter choice rule.

3 Variational regularisation

This question deals with basic concepts of convex analysis and variational regularisation methods.

1. Write down the definition of the *convex conjugate* E^* for a proper, lower semi-continuous and convex functional E.

2. Compute the convex conjugates of the following functions or functionals:

   (a) $E : \mathbb{R} \to \mathbb{R}, \quad E(x) := |x|$.

   (b) $E : \mathbb{R} \to \mathbb{R}_\infty, \quad E(x) := \chi_{[-1,1]}(x) + \frac{1}{2}|x|^2$, with $\chi_{[-1,1]}(x) := \begin{cases} 0 & \text{if } |x| \leqslant 1 \\ \infty & \text{else} \end{cases}$.

3. In a finite dimensional space, boundedness of a sequence implies that the sequence has a strongly convergent subsequence. What is the analogue for an infinite-dimensional Hilbert space? State similar statements for reflexive and non-reflexive Banach spaces.

4. Write down the definition of the *proximal operator* prox_E for a convex functional E.

5. Compute a simple formula for the solution of the proximal operator for the convex functional $E : \mathcal{X} \to \mathbb{R}_\infty, \quad E(x) := \alpha J(cx - y) + \langle x, z \rangle$, for $\alpha > 0$, $c \in \mathbb{R}$, $y \in \mathcal{X}$, $z \in \mathcal{X}^*$, $\mathcal{X}$ being a Banach space and $J : \mathcal{X} \to \mathbb{R}_\infty$ being a proper, lower semi-continuous and convex functional.

6. Verify

$$p \in \partial J(u) \qquad \Leftrightarrow \qquad u \in \partial J^*(p)$$

   for a proper, lower semi-continuous and convex functional $J : \mathcal{U} \to \mathbb{R}_\infty$ and its convex conjugate $J^* : \mathcal{U}^* \to \mathbb{R}_\infty$, for $\mathcal{U}$ being a Hilbert space.
   Hint: Prove the equivalence $p \in \partial J(u) \Leftrightarrow J(u) + J^*(p) = \langle u, p \rangle$ first. You may exploit the fact that under the stated assumptions $J = J^{**}$ holds true.

7. Prove Moreau's identity, respectively Moreau's decomposition, which states

$$u = \text{prox}_J(u) + \text{prox}_{J^*}(u)\,,$$

   for all $u \in \mathcal{U}$ and $J : \mathcal{U} \to \mathbb{R}_\infty$ as defined in Exercise 6.

4 Bregman distances

This question deals with numerous aspects of generalised Bregman distances.

1. Write down the definitions of the *subdifferential* for convex functionals and the *Bregman distance* as well as the *symmetric Bregman distance.*

2. Let $\mathcal{U}$ be a Banach space and $\mathcal{V}$ be a Hilbert space. Verify the error estimate

$$D_J^{\mathrm{symm}}(u_\alpha, u^\dagger) \leqslant \frac{\delta^2}{2\alpha} + \frac{\alpha \|w\|_{\mathcal{V}}^2}{2} \,,$$

for $\alpha > 0$, $J : \mathcal{U} \to \mathbb{R}_\infty$ being a proper, lower semi-continuous and convex functional, and u_α defined as

$$u_\alpha := \arg\min_{u \in \mathcal{U}} \left\{ \frac{1}{2}\|Ku - f^\delta\|_{\mathcal{V}}^2 + \alpha J(u) \right\} \,, \tag{1}$$

and $f := Ku^\dagger$ with $\|f - f^\delta\|_{\mathcal{V}} \leqslant \delta$ for some $u^\dagger$ that satisfies the source condition $K^* w \in \partial J(u^\dagger)$. *Hint: Exploit the optimality conditions of* (1) *to prove this – compared to the lecture – stronger estimate.*

3. Prove that the weighted one-norm $J : \ell^2 \to \mathbb{R}_\infty$ with

$$J(u) := \begin{cases} \sum_{k=1}^\infty w_k |u_k| & u \in \ell^1 \\ \infty & u \in \ell^2 \setminus \ell^1 \end{cases} \tag{2}$$

and weights that satisfy $0 < c \leqslant w_k < \infty$, for all $k \in \mathbb{N}$, is lower semi-continuous, and compute the corresponding Bregman distance.

4. Show that for $\beta > 0$ the elastic net, i.e. $R_\alpha f^\delta := \arg\min_{u \in \ell^2} \left\{ \frac{1}{2}\|Ku - f^\delta\|_{\ell^2}^2 + \alpha J(u) \right\}$ with

$$J(u) := \begin{cases} \|u\|_{\ell^1} + \beta \|u\|_{\ell^2}^2 & u \in \ell^1 \\ \infty & u \in \ell^2 \setminus \ell^1 \end{cases} \,, \tag{3}$$

for $\alpha > 0$ and $f^\delta \in \ell^2$, is a convergent regularisation (in the norm sense for Hilbert spaces) and specify a suitable parameter choice rule. Further show that R_α is a non-linear operator.

5. Assume that u_λ is a generalised singular vector, i.e. we have $\|Ku_\lambda\|_{\mathcal{V}} = 1$, $\lambda K^* K u_\lambda \in \partial J(u_\lambda)$ and $\lambda = J(u_\lambda)$, for a linear operator $K : \mathcal{U} \to \mathcal{V}$, mapping between a Banach space $\mathcal{U}$ and a Hilbert space $\mathcal{V}$, and a proper, convex, lower semi-continuous and absolutely one-homogeneous functional $J : \mathcal{U} \to \mathbb{R}_\infty$. Further assume that we have data $f = \gamma K u_\lambda$ for $\gamma > 0$. Show that for fixed $\alpha > 0$ the iterates of the Bregman iteration, i.e. for $k = 1, 2, \dots$

$$\begin{aligned} u_\alpha^{k+1} &\in \arg\min_{u \in \mathcal{U}} \left\{ \frac{1}{2}\|Ku - f\|_{\mathcal{V}}^2 + \alpha D_J^{p_\alpha^k}(u, u_\alpha^k) \right\} \\ p_\alpha^{k+1} &= p_\alpha^k + \frac{1}{\alpha} K^*(f - K u_\alpha^{k+1}) \end{aligned} \,, \tag{4}$$

with $u_\alpha^0 = p_\alpha^0 = 0$ and $p_\alpha^k \in \partial J(u_\alpha^k)$ for all $k \in \mathbb{N}$, converge to $u_\alpha^{k_*} = \gamma u_\lambda$ after a finite number of iterations k_*.

END OF PAPER