UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 12 June, 2017 9:00 am to 11:00 am

PAPER 322

BINARY STARS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A galaxy is 10 Gyr old and has been forming stars at a constant rate since its birth. Show that the fraction of stars younger than t is

$$Y(t) = 0.1 \left(\frac{t}{\mathrm{Gyr}} \right).$$

Stars in the galaxy form with masses M chosen from an initial mass function

$$n(M)dM = \begin{cases} k \left(\frac{M_\odot}{M} \right)^2 dM, & M > 0.1\, M_\odot, \\ 0, & M < 0.1\, M_\odot, \end{cases}$$

where $n(M)dM$ is the number of stars with masses between M and $M + dM$ and k is a constant. Show that the fraction X of stars that form with mass greater than M is

$$X(M) = \frac{0.1}{(M/M_\odot)}, \quad M > 0.1\, M_\odot.$$

A star of mass M spends $9\,\mathrm{Gyr}/(M/M_\odot)$ on the main sequence and then a further $1\,\mathrm{Gyr}/(M/M_\odot)$ as a red giant before becoming a white dwarf. On a sketch of the (M, t) plane indicate the area that contains stars which are currently giants, giving formulae for its boundaries. Show that this maps to a triangle in the (X, Y) plane and hence that $1/180$ of stars are currently giants and that $1/20$ are currently white dwarfs.

All stars in the galaxy actually form in binary systems with the masses of the two components chosen independently from the above distribution. By considering the subdivisions of a cube, or otherwise, find

(i) the fraction of systems contain a red giant,

(ii) the fraction of systems containing a red giant also contain another evolved star, either another red giant or a white dwarf and show that

(iii) for every binary system containing two red giants there are 18 containing both a white dwarf and a red giant and 81 containing two white dwarfs.

A second galaxy is similar to the first except that all stars formed in a single burst at a time $0 < t < 10\,\mathrm{Gyr}$ ago instead of continuously. What fraction of systems consist of two red giants now? For what range of starburst age t does this exceed the fraction of red giants seen now in the first galaxy?

2

A cataclysmic variable consists of a white dwarf of mass M_1 and a low-mass main-sequence star of mass M_2 in a circular orbit with separation a. The main-sequence star is filling its Roche lobe, of radius $R_{\rm L}$, and transferring mass to the white dwarf at a rate $\dot{M}_1 = -\dot{M}_2$. The mass ratio $q = M_2/M_1 < 1$. The hydrostatic and thermal equilibrium radius of the main-sequence star can be approximated by

$$\frac{R_2}{R_\odot} = \frac{M_2}{M_\odot}$$

while, for a suitable range of mass ratios, the Roche-lobe radius $R_{\rm L}$ obeys

$$\frac{R_{\rm L}}{a} = 0.46\left(\frac{M_2}{M}\right)^{\frac{1}{3}},$$

where $M = M_1 + M_2$. Assuming that the main-sequence star remains in thermal equilibrium, show that the period P of the binary is given by

$$\frac{P}{P_0} = \frac{M_2}{M_\odot}$$

for some constant P_0.

Neglecting spin angular momentum find $\dot{R}_{\rm L}/R_{\rm L}$ as a function of $\dot{M}_2/M_2$ when $\dot{J} = 0$ and compare it with $\dot{R}_2/R_2$. Why would the mass transfer be dynamically unstable if $q > 4/3$?

Describe briefly the mechanisms that can lead to angular momentum loss $\dot{J} < 0$ and maintain mass transfer if $q < 4/3$.

In a classical nova, once a layer of hydrogen-rich material of mass $\delta m \approx 10^{-4}\,M_\odot$ has accumulated on the surface of the white dwarf thermonuclear reactions ignite in the degenerate material. These expel the entire layer of mass δm isotropically from the system in the nova explosion over a time that is very short compared with the mass transfer timescale but very long relative to P. Show that the change in separation $\delta a/a = \delta m/M$ and that the change in Roche lobe radius $\delta R_{\rm L}/R_{\rm L} = 4\delta m/3M$, assuming $\delta m \ll M$.

Deduce that the mass transfer ceases. Assuming that there is a constant rate of angular momentum loss $-\dot{J}$ until mass transfer is resumed and that this rate remains the same until the next nova explosion show, again to first order in $\delta m/M$, that the ratio of the time spent detached $t_{\rm d}$, during this interruption, to the time spent semi-detached $t_{\rm s}$, while the hydrogen-rich layer is accumulating, is

$$\frac{t_{\rm d}}{t_{\rm s}} = \frac{2q}{(4-3q)(1+q)}.$$

3

A binary star consists of two point-like objects of masses M_1 and M_2 in an elliptical orbit with semi-major axis a and eccentricity e. From Newton's law of gravity show that the separation of the stars $\boldsymbol{r}$, evolves according to

$$\ddot{\boldsymbol{r}} = -\frac{GM}{r^3}\boldsymbol{r},$$

where $M = M_1 + M_2$ and $r = |\boldsymbol{r}|$.

In the absence of any perturbation show that the orbital energy E, angular momentum $\boldsymbol{J}$ and eccentricity e remain constant.

Show that at $\boldsymbol{x}$, far from the centre of mass of the binary star, its combined gravitational potential can be expanded to second order in its separation $\boldsymbol{r}$ as

$$\phi(\boldsymbol{x}) = -\frac{GM}{x} - Gq_{ij}l_{ij}(\boldsymbol{x}),$$

where

$$q_{ij} = \frac{\mu}{2}(3r_ir_j - r^2\delta_{ij}),$$

$$l_{ij}(\boldsymbol{x}) = \frac{3x_ix_j - x^2\delta_{ij}}{3x^5},$$

$\mu = M_1M_2/M$, $r = |\boldsymbol{r}|$ and $x = |\boldsymbol{x}|$.

In the weak field limit for general relativity

$$\dot{E} = -\frac{4G}{45c^5}\frac{d^3q_{ij}}{dt^3}\frac{d^3q_{ij}}{dt^3}.$$

Show that, instantaneously,

$$\dot{E} = -\frac{32G^3M^2\mu^2}{5c^5r^4}\left(\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} - \frac{11}{12}\dot{r}^2\right),$$

where $\dot{r} = dr/dt$.

Explain why a circular orbit is expected to remain circular and show that, for such an orbit of separation a,

$$\frac{\dot{a}}{a} = -\frac{64G^3M^2\mu}{5c^5a^4}.$$

Comment on the consequences for the evolution of close binary stars.

END OF PAPER