UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 9 June, 2017 1:30 pm to 3:30 pm

PAPER 321

DYNAMICS OF ASTROPHYSICAL DISCS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Viscous evolution and vertical structure

(a) A young protoplanetary disk is receiving mass from the interstellar medium. The equation governing its evolution is

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{2\pi r}\partial_r \mathcal{F} + S(r),$$

where Σ is the disk surface density, $S(r)$ is the steady supply rate from the disk's environment, and the radial mass flux $\mathcal{F}$ is defined by

$$\mathcal{F} = 6\pi r^{1/2}\partial_r \left(r^{1/2}\overline{\nu}\Sigma\right),$$

with $\overline{\nu}$ the mean turbulent viscosity. The viscous torque vanishes at the disk's inner radius $r = r_1$, but there is no radial mass flux at the outer radius $r = r_2$.

(i) Show that the steady state surface density profile is

$$\Sigma = \frac{1}{3\overline{\nu} r^{1/2}} \int_{r_1}^{r} u^{-1/2} \left(\int_{u}^{r_2} v\, S(v) dv \right) du.$$

(ii) Suppose the mass supply is localised entirely at the outer radius, so that $S = S_0\delta(r - r_2)$, where δ is the Dirac delta function and S_0 a constant. Calculate and comment on the resulting profile for Σ.

(b) In standard notation, the vertical structure of an accretion disc is described by the following equations

$$\frac{dP}{dz} = -\rho\Omega^2 z, \qquad \frac{dF}{dz} = \frac{9}{4}\mu\Omega^2,$$
$$\frac{dT}{dz} = -\frac{3\kappa\rho F}{16\sigma T^3}, \qquad P = \frac{k\rho T}{\mu_m m_p}.$$

(i) Suppose that $\mu = F = 0$ and $P\rho^{-\gamma}$ is a constant. Solve for the disk's vertical structure, showing that

$$\rho = \rho_m \left(1 - \frac{z^2}{H^2}\right)^{1/(\gamma-1)},$$

where ρ_m is the density at the midplane, and H is the disk's semi-thickness. Give an expression for H in terms of the constant parameters of the disk.

(ii) Consider a dwarf nova disk in which $\mu \neq 0$ and $F \neq 0$ and where the opacity is dominated by negative Hydrogen ions. In this regime we have the approximation

$$\kappa = \kappa_0 \rho^{1/3} T^{10},$$

with κ_0 a constant. Assuming that $\mu = \alpha P/\Omega$, where α is a constant, employ an order of magnitude approach to show that

$$H^{41/3} \sim \left(\frac{\sigma}{\kappa_0}\right)\left(\frac{\mu_m m_p}{k}\right)^{-6}\Omega^{-15}\alpha^{-1}\Sigma^{-7/3},$$

where $\Sigma \sim \rho H$ is the disk's surface density. Obtain an estimate for the angular thickness H/r and comment on the flaring of the disk. How does the sound speed vary with radius?

(iv) By observing how the cooling and heating rates depend on T, give a qualitative argument for why the dwarf nova disk is thermally unstable in this regime.

2

The vertical shear instability

Strongly irradiated disks possess a rotation profile that depends on z, in addition to radius. A small region of a disk exhibiting vertical shear can be described by the following shearing sheet model

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho_0}\nabla P - 2\Omega\,\mathbf{e}_z\times\mathbf{u} + 3\Omega^2 x\,\mathbf{e}_x - 2\Omega^2\,q\,z\,\mathbf{e}_x,$$
$$\nabla\cdot\mathbf{u} = 0$$

where $\mathbf{u}$, P, and ρ_0 are the velocity, pressure, and (constant) density, respectively. Ω is the constant rotation frequency at the origin of the sheet, and q is a small dimensionless constant.

(a) Derive the energy equation in the form

$$\partial_t\left(\tfrac{1}{2}\rho_0|\mathbf{u}|^2\right) + \nabla\cdot\mathbf{F} = S,$$

where $\mathbf{F}$ and S are to be determined. Does the model conserve energy?

(b) Show that the governing equations admit the following steady solution

$$\mathbf{u} = -\tfrac{1}{2}\,(3x - 2qz)\,\Omega\,\mathbf{e}_y, \qquad P = P_0,$$

where P_0 is a constant. Describe in words what this solution corresponds to.

(c) Consider perturbations to this equilibrium of the form

$$\mathbf{u}' = \tilde{\mathbf{u}}f(\xi)\mathrm{e}^{st}, \qquad P' = \tilde{P}g(\xi)\mathrm{e}^{st},$$

where $\tilde{\mathbf{u}}$ and $\tilde{P}$ are constants, s is a growth rate, and f and g are non-constant functions of the variable $\xi = k_x x + k_z z$, for k_x and k_z constant wavenumbers.

Using incompressibility, show that $\mathbf{u}'\cdot\nabla\mathbf{u}' = 0$.

What relationship must hold between g and f for the perturbations to be solutions of the governing equations? Show that $\tilde{\mathbf{u}}$ and $\tilde{P}$ satisfy

$$s\tilde{u}_x = -\frac{1}{\rho_0}k_x\tilde{P} + 2\Omega\tilde{u}_y, \qquad s\tilde{u}_y = -\frac{1}{2}\Omega\tilde{u}_x - q\Omega\tilde{u}_z,$$
$$s\tilde{u}_z = -\frac{1}{\rho_0}k_z\tilde{P}, \qquad k_x\tilde{u}_x + k_z\tilde{u}_z = 0.$$

(d) Derive the dispersion relation for these perturbations:

$$s^2 = -\frac{k_z^2}{k^2}\Omega^2\left(1 - 2q\frac{k_x}{k_z}\right).$$

What is the instability criterion?

Show that the maximum growth rate is Ωq, in the limit $0 < q \ll 1$.

3

Density waves and dust

(a) The equations of a razor-thin compressible disk in the shearing sheet are

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\Sigma}\nabla P - 2\Omega\mathbf{e}_z \times \mathbf{u} - \nabla\Phi_t,$$
$$\partial_t \Sigma + \mathbf{u}\cdot\nabla\Sigma = -\Sigma\nabla\cdot\mathbf{u},$$

where $\mathbf{u}$, Σ, and P are the velocity, surface density, and pressure. The tidal potential is $\Phi_t = -(3/2)\Omega^2 x^2$, and the fluid is assumed barotropic, $P = P(\rho)$.

(i) Write down the linearised equations governing small axisymmetric perturbations to the equilibrium: $\Sigma = \Sigma_0 =$ constant, $\mathbf{u} = \mathbf{u}_0 = -(3/2)\Omega\, x\, \mathbf{e}_y$.

(ii) Assume that the perturbations are $\propto \exp(\mathrm{i}kx - \mathrm{i}\omega t)$, where k is the radial wavenumber and ω is a wave frequency. Derive the dispersion relation for density waves in the shearing sheet and give a physical explanation for the various terms in this expression.

(iii) In the limit of long wavelengths, show that the real part of the velocity components of the density wave (minus the background shear) may be approximated by

$$u_x' = u\cos(kx - \Omega t), \qquad u_y' = \tfrac{1}{2}u\sin(kx - \Omega t), \tag{†}$$

where u is an arbitrary real constant.

(b) Suppose dust particles are distributed uniformly throughout the gas. We can model the dust in the shearing sheet as a pressureless fluid via the following equation

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -2\Omega\mathbf{e}_z \times \mathbf{v} - \nabla\Phi_t - \epsilon\Omega(\mathbf{v} - \mathbf{u}),$$

where $\mathbf{v}$ denotes the velocity of the dust fluid, ϵ is a constant drag coefficient, and $\mathbf{u}$ is the velocity of the gas.

When $\mathbf{u} = \mathbf{u}_0$ (given above) the associated steady solution in the dust is $\mathbf{v} = \mathbf{v}_0 = \mathbf{u}_0$. When a density wave passes through the gas, the dust is agitated and $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}'$, where $\mathbf{v}'$ is the perturbation induced by the density wave.

(i) Set $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}'$, using (†). Assume the dust perturbation $\mathbf{v}'$ is small, and suppose the dust is only weakly coupled to the gas, so that $0 < \epsilon \ll 1$. Derive the following equation for v_x'

$$\partial_t^2 v_x' + 2\epsilon\Omega\,\partial_t v_x' + (1+\epsilon^2)\Omega^2\, v_x' = 2\Omega^2\epsilon u\sin(kx - \Omega t),$$

which describes a forced and damped oscillator.

(ii) By considering this equation's complementary function and particular integral, show that at late times the dust fluid supports density waves oscillating in phase with the gas and at the same amplitude.

(iii) The dust surface density Σ_d obeys the equation

$$\partial_t \Sigma_d = -\nabla\cdot(\Sigma_d \mathbf{v}).$$

At late times, what is the maximum fractional concentration of the dust in the crests of the density waves?

END OF PAPER