UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 6 June, 2017 9:00 am to 12:00 pm

PAPER 320

GALACTIC ASTRONOMY AND DYNAMICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) What is meant by the *relaxation time* of a stellar system?

Consider a system of N stars of mass m with typical speed v moving in a spherical volume of radius R. Show the minimum impact parameter $b_{\rm min}$ for which the impulse approximation holds good for describing the effect of encounters is

$$b_{\rm min} \approx \frac{Gm}{v^2}.$$

Let the number density of stars be n. Show that the relaxation time is

$$t_{\rm rel} \approx \left(\frac{3}{32\pi}\right) \frac{v^3}{G^2 m^2 n \log \Lambda},$$

where $\Lambda = R/b_{\rm min}$.

Hence, show that

$$t_{\rm rel} \approx \frac{N}{8 \log N} t_{\rm dyn},$$

where $t_{\rm dyn}$ is the crossing time.

What is the smallest number of particles that an N-body simulator should use to ensure that the simulations of round galaxies are collisionless?

(b) Relaxation effects in a razor-thin disks are different. Stars are restricted to move in the plane. Disks are rotationally supported, and so stars typically pass each other at speeds that are a small fraction of the circular speed, say λv, where $\lambda \approx 0.1$. Show that in a time T, a star confined to a razor-thin disk will suffer dn encounters with other stars with impact parameters between b and $b + db$, where

$$dn \approx 2n\lambda v T db,$$

and n is the two-dimensional number density.

Hence, derive the relaxation time as

$$t_{\rm rel} \approx \frac{\lambda^3 v^3 b_{\rm min}}{8 G^2 m^2 n}.$$

By using the virial theorem, show that

$$t_{\rm rel} \approx \lambda t_{\rm dyn}.$$

Are infinitesimally thin disks collisionless?

(c) Real galactic disks are not infinitesimally thin, but have some characteristic thickness z_0. The formulae for relaxation in a spherical system is now shorter by a factor

$$(\lambda^3) \left(\frac{z_0}{R}\right) \left(\frac{\log[R/b_{\rm min}]}{\log[z_0/b_{\rm min}]}\right).$$

Without doing any detailed calculations, explain the physical origin of each of the three terms in round brackets.

What is the smallest number of particles that an N-body simulator should use to ensure that the simulations of flattened disk galaxies are collisionless?

2

Show that the binding energy per unit mass E and the absolute value of the angular momentum per unit mass L are integrals of motion for a star moving in a spherical potential ψ (with the convention that the gravitational force $= \nabla\psi$).

Suppose that the distribution function f is isotropic and so depends on E only. Derive Eddington's formula

$$f(E) = \frac{1}{\sqrt{8}\pi^2} \frac{d}{dE} \int_0^E \frac{d\psi}{(E-\psi)^{1/2}} \frac{d\rho}{d\psi},$$

where ρ is considered as a function of ψ.

Show that the second velocity moments are all equal.

Now suppose we seek an anisotropic distribution function of form $f(E, L) = f_E(E)/L$. Show that

$$f_E(E) = \frac{1}{2\pi^2} \frac{d(r\rho)}{d\psi}\bigg|_{\psi=E},$$

where $r\rho$ is considered as a function of ψ.

Suppose that the radius of the largest orbit with energy E is r_E. Show that the distribution function can be written as

$$f(E, L) = \frac{g(r_E)}{2\pi^2 L},$$

where

$$g(r_E) = \frac{\rho + d\rho/d\psi}{d\psi/dr}\bigg|_{r=r_E}.$$

The anisotropy parameter β is defined as

$$\beta = 1 - \frac{\langle v_\theta^2 \rangle + \langle v_\phi^2 \rangle}{2\langle v_r^2 \rangle},$$

where angled brackets denote averages over the distribution function. Show that $\beta = 1/2$, and comment on the implications of your result.

The Hernquist model has potential

$$\psi = \frac{GM}{r+a},$$

where M is the mass of the model and a is a scalelength. Find the density and show that there is an anisotropic distribution function of the form

$$f(E, L) = \frac{3a}{4\pi^3} \frac{E^2}{G^3 M^2 L}.$$

3

(a) Give a physical explanation of the Jeans instability for a homogeneous fluid system with density ρ_0 and sound speed v_s, and show that perturbations with a scale longer than $v_\mathrm{s}/\sqrt{G\rho_0}$ are unstable.

The continuity and momentum equations for a fluid with density ρ, gravitational potential ϕ and pressure p are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0,$$

and

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\frac{1}{\rho}\nabla p - \nabla \phi.$$

Explaining carefully any assumptions you make, derive the dispersion relation

$$\omega^2 - v_\mathrm{s}^2 k^2 + 4\pi G \rho_0 = 0,$$

for perturbations with wavenumber k and angular frequency ω.

Hence, show that the system is unstable to perturbations less than some critical wavenumber k_J.

Now suppose the gravitational potential due to a body of mass m is modified from the Newtonian form to the Yukawa potential $\phi = -Gm\exp(-\alpha r)/r$, where α is a constant. Show that the dispersion relation becomes

$$\omega^2 - v_\mathrm{s}^2 k^2 + 4\pi G \rho_0 \frac{k^2}{k^2 + \alpha^2} = 0.$$

Derive the necessary condition for instability.

Find the range of unstable wavenumbers and give a physical explanation of your result.

(b) For an infinite, homogeneous stellar system with density ρ_0 and with equilibrium distribution function $f_0(\underline{v})$, show that the dispersion relation is

$$1 + \frac{4\pi G}{k^2} \int d^3 v \, \frac{\underline{k} \cdot \partial f_0 / \partial \underline{v}}{\underline{k} \cdot \underline{v} - \omega} = 0.$$

Now suppose the stellar system has an equilibrium distribution function of the form (the Cauchy distribution):

$$f_0(\underline{v}) = \frac{\rho_0 b}{\pi^2} \frac{1}{(v^2 + b^2)^2},$$

where b is a constant. What is the Jeans wavenumber?

Hint: You are reminded of the standard integral ($a > 0, n > 1/2$)

$$\int_{-\infty}^{\infty} \frac{dX}{(X^2 + a^2)^n} = \frac{\Gamma(n - 1/2)\sqrt{\pi}}{\Gamma(n)} \frac{1}{a^{2n-1}},$$

where $\Gamma()$ is the Gamma function.

4 A star has the Lagrangian $L(q_j, \dot{q}_j, t)$, where the q_j are generalized coordinates and the $\dot{q}_j$ are the corresponding generalized velocities. Show how to define a Hamiltonian $H(q_j, p_j)$ where the p_j are canonical momenta conjugate to the q_j.

Derive Hamilton's equations.

Let (R, ϕ, z) be the familiar cylindrical polar coordinate system. Spheroidal coordinates (λ, μ) are defined as the roots for τ of the equation

$$\frac{R^2}{\tau+\alpha} + \frac{z^2}{\tau+\beta} = 1,$$

where $-\beta \leqslant \mu \leqslant -\alpha \leqslant \lambda$ and α and β are positive constants. Together with the azimuthal angle ϕ, this provides an orthogonal curvilinear coordinate system (λ, μ, ϕ). The scale-factors of the coordinates are

$$\begin{aligned} h_\lambda^2 &= \frac{\lambda-\mu}{4(\lambda+\alpha)(\lambda+\beta)}, \\ h_\mu^2 &= \frac{\mu-\lambda}{4(\mu+\alpha)(\mu+\beta)}, \\ h_\phi^2 &= \frac{(\lambda+\alpha)(\mu+\alpha)}{\alpha-\beta}. \end{aligned}$$

Show that the Hamiltonian of a star moving in the potential $\psi(\lambda, \mu, \phi)$ is

$$H = \frac{1}{2}\left(\frac{p_\lambda^2}{h_\lambda^2} + \frac{p_\mu^2}{h_\mu^2} + \frac{p_\phi^2}{h_\phi^2}\right) - \psi(\lambda, \mu, \phi),$$

where $(p_\lambda, p_\mu, p_\phi)$ are canonical momenta conjugate to (λ, μ, ϕ), and we are using the convention that the gravitational force is $\nabla\psi$.

If the potential has the form (the Stäckel potential)

$$\psi(\lambda, \mu) = \frac{G(\lambda) - G(\mu)}{\lambda - \mu},$$

show by separation of the Hamilton-Jacobi equation that there are three integrals of motion, namely the energy E, the angular momentum parallel to the symmetry axis $p_\phi = Rv_\phi$ and I, where

$$I = \frac{1}{2}\Big(\mu v_\lambda^2 + \lambda v_\mu^2 + (\lambda + \mu + \alpha)v_\phi^2\Big) - \frac{\mu G(\lambda) - \lambda G(\mu)}{\lambda - \mu}.$$

Suppose a steady-state stellar system has a Stäckel potential. By using Jeans Theorem, show that the mixed velocity second moments vanish

$$\langle v_\lambda v_\mu \rangle = \langle v_\lambda v_\phi \rangle = \langle v_\mu v_\phi \rangle = 0,$$

where angled brackets denote averages over the distribution function.

Describe qualitatively the form of the stellar orbits in the model. Your answer should include a diagram that shows the volume occupied by a typical stellar orbit together with the coordinate surfaces.

END OF PAPER