UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 5 June, 2017 1:30 pm to 4:30 pm

PAPER 316

PLANETARY SYSTEM DYNAMICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) A comet of mass m and density ρ is on an orbit with semimajor axis a and eccentricity e around a star of mass $M_\star \gg m$, luminosity $L_\star$ and radius $R_\star$. Derive the constants of motion for the two body problem to show that the specific angular momentum of the comet's orbit is given by $h = \sqrt{\mu a(1-e^2)}$, and its orbital velocity by $v = \sqrt{\mu(2/r - 1/a)}$, where $\mu = G(M_\star + m)$.

(ii) Show that when the comet is at a distance r from the star, its radial velocity is

$$v_{\rm r} = v_{\rm k}(a)\sqrt{2u - 1 - u^2(1-e^2)},$$

where $v_{\rm k}(a) = \sqrt{\mu/a}$ is the Keplerian orbital velocity at a distance a, and $u = a/r$.

(iii) The comet disintegrates into fragments with diameters from $D_{\rm min}$ to $D_{\rm max} \gg D_{\rm min}$. If the number of fragments in the size range D to $D + dD$ is $n(D)dD$, where $n(D) \propto D^{-\alpha}$ and $3 < \alpha < 4$, show that the total cross-sectional area of the fragments is

$$\sigma_{\rm tot} \approx \left(\frac{3(4-\alpha)m}{2(\alpha-3)\rho}\right) D_{\rm min}^{3-\alpha} D_{\rm max}^{\alpha-4}.$$

(iv) Over time the fragments become evenly distributed around the orbit so that they form a wire with a line density at each point that is proportional to the orbital velocity. If the particles act like black bodies (and so absorb and then re-emit all of the starlight they intercept), show that the fractional luminosity of the fragments, i.e., the ratio of their thermal luminosity to that of the star, $f = L_{\rm th}/L_\star$, is given by

$$f = \sigma_{\rm tot}/[4\pi a^2\sqrt{1-e^2}],$$

for which you may find it useful that $d[\sin^{-1}(x/b)]/dx = (b^2 - x^2)^{-1/2}$.

(v) Derive an expression for the size of comet that would need to be broken up to create a given fractional luminosity of f.

(vi) Assume now that the comet's orbital plane is aligned to our line of sight so that the wire covers the stellar surface near its equator. Show that the rate at which cross-sectional area crosses the line of sight to the star is $\dot{\sigma} = 2fh$.

(vii) Hence show that the fraction by which the star appears fainter because of the material in front of it is given by $(4/\pi)(r/R_\star)f$, where r is the distance from the star to the fragments in the direction of our line of sight.

2

(i) Consider a dust particle of mass m and cross-sectional area σ that is in orbit around a star of mass $M_\star$ and luminosity $L_\star$, where a, e and ϖ are the semimajor axis, eccentricity and longitude of pericentre respectively of the orbit. The particle is subject to a perturbing acceleration of $\bar{R}\hat{\mathbf{r}} + \bar{T}\hat{\boldsymbol{\theta}}$, where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are orthogonal unit vectors in the orbital plane where $\hat{\mathbf{r}}$ is in the direction from the star to the particle. Show that the semimajor axis of the particle's orbit changes at a rate

$$\dot{a} = 2\sqrt{\frac{a^3}{\mu(1-e^2)}}[\bar{R}e\sin f + \bar{T}(1+e\cos f)],$$

where $\mu = GM_\star$ and f is the true anomaly of the particle in its orbit.

(ii) Give a physical explanation for Poynting-Robertson (P-R) drag and for black body dust describe how this results in a force of

$$\mathbf{F}_{\rm pr} = m\left(\frac{\beta\mu}{r^2}\right)\left[-2\left(\frac{\dot{r}}{c}\right)\hat{\mathbf{r}} - \left(\frac{r\dot{\theta}}{c}\right)\hat{\boldsymbol{\theta}}\right],$$

where r is the distance of the particle from the star, θ is the azimuthal angle of the particle in the orbital plane, c is the speed of light, and the dependence of β on other physical parameters should be given.

(iii) Show that P-R drag causes this circumstellar orbit to decay at a rate when averaged around the particle's orbit of

$$\dot{a}_{\rm cs} = -\left(\frac{\beta\mu}{c}\right)\frac{1}{a}\frac{2+3e^2}{(1-e^2)^{3/2}}.$$

(iv) Consider now that the dust particle is on a circular orbit at a distance a from a planet of mass $M_{\rm p}$ and radius $R_{\rm p}$, which is itself on a circular orbit around the star at a distance $a_{\rm p}$. The planet is much less massive than the star so that $(M_{\rm p}/M_\star)^{1/3} \ll 1$. Show that P-R drag from the stellar radiation causes the particle's circumplanetary orbit to decay at a rate when averaged around the particle's orbit of

$$\dot{a}_{\rm cp} = -3\left(\frac{\beta\mu}{c}\right)\frac{a}{a_{\rm p}^2}.$$

(v) Derive and compare the timescales for P-R drag to make a particle on a circular circumstellar orbit at a distance $a_{\rm p}$ reach the star with that of a particle on a circumplanetary orbit initially at a distance a_0 from a planet at $a_{\rm p}$ from the star to reach the planet's surface.

(vi) Comment on what other physical mechanisms might act to remove particles that are on circumstellar or circumplanetary orbits.

3

(i) A coplanar planetary system is comprised of two planets b and c in $(p+q):p$ mean motion resonance, where p and q are integers. The innermost planet b is on a circular orbit, while the outermost planet c's orbit is elliptical with pericentre at a longitude ϖ_{c}; the longitudes of the planets at any given time are λ_{b} and λ_{c}. State which resonant angle ϕ_{bc} is expected to be librating and why.

(ii) Describe how the longitudes at which conjunctions between b and c occur depend on ϕ_{bc}, ϖ_{c}, p and q.

(iii) Give a physical explanation for the expected change in ϕ_{bc} following a conjunction, and so the value about which ϕ_{bc} would be expected to undergo stable libration for $q = 1$, 2 or 3.

(iv) The same system is found to have an additional outer planet d which is on a circular coplanar orbit in $(m+n):m$ mean motion resonance with planet c, where m and n are integers. Provide a description, similar to that above, for the longitudes at which conjunctions occur, and for the expected libration of the resonant angle ϕ_{cd} for $n = 1$, 2 or 3.

(v) Hence explain which combinations of p, q, m and n might potentially result in the three planets being seen to transit in front of the star at the same time.

(vi) Consider now that the ratio of the orbital periods of planets b and c are offset from the exact resonance so that $P_{\mathrm{c}}/P_{\mathrm{b}} = (1+\delta)(p+q)/p$, where $\delta \ll 1$. If planet c is observed to transit the star a time τ after planet b, show that the longitude at which they next have a conjunction relative to our line of sight occurs at

$$\Lambda_{\mathrm{bc}} = 2\pi \left(1 - \frac{\tau}{P_{\mathrm{b}}}\right) \left(\frac{p}{q}\right) \left[1 + \delta \frac{(p+q)}{q}\right]^{-1}.$$

(vii) Hence show that the rate of precession of conjunctions near this longitude is

$$\dot{\Lambda}_{\mathrm{bc}} = -2\pi\delta(p/q)/P_{\mathrm{b}},$$

and comment on the implications for the possibility of observing the three planets to transit at the same time.

4

(i) Derive the escape velocity $v_{\rm esc}$ for a comet of mass M and radius R from the surface of a planet of mass $M_{\rm p} \gg M$ and radius $R_{\rm p}$.

(ii) Consider a situation in which the comet approaches the planet with a relative velocity at large separation of v_∞, then undergoes a hyperbolic encounter with the planet. Show that the angle θ through which the comet's motion is deflected by the encounter is given by

$$\sin(\theta/2) = [1 + r_{\rm min} v_\infty^2/\mu]^{-1},$$

where $\mu = GM_{\rm p}$ and $r_{\rm min}$ is the minimum separation between the two objects during the encounter.

(iii) Explain why the magnitude of the change in the comet's velocity vector has a maximum possible value and show that this is given by $v_{\rm esc}/\sqrt{2}$.

(iv) Consider now that the planet and comet are orbiting a star of mass $M_\star$, and that the planet is on a circular orbit of radius $a_{\rm p}$. Give a physical explanation for the importance of the ratio of a planet's escape velocity $v_{\rm esc}$ to its Keplerian orbital speed $v_{\rm k}$ around the star in determining the eventual fate of comets being scattered by the planet.

(v) Show that

$$M_{\rm p} \propto (v_{\rm esc}/v_{\rm k})^3 M_\star^{3/2} a_{\rm p}^{-3/2} \rho_{\rm p}^{-1/2},$$

where $\rho_{\rm p}$ is the planet's density, and give the constant of proportionality.

(vi) For a comet undergoing repeated encounters with a planet, its orbit can be considered to undergo diffusion in energy with a characteristic diffusion coefficient for the inverse of its semimajor axis of $(10/a_{\rm p})(M_{\rm p}/M_\star)$. Determine the characteristic timescale for comets undergoing diffusion to be ejected.

(vii) Show how the above results can be used to predict the most likely eventual outcome for comets undergoing multiple scattering events with a planet in different regions of a plot of $M_{\rm p}$ versus $a_{\rm p}$ for a star of given age $t_\star$.

(viii) Comment on other factors which might affect the outcome of scattering of comets by planets.

END OF PAPER