UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 1 June, 2017 1:30 pm to 4:30 pm

PAPER 314

ASTROPHYSICAL FLUID DYNAMICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{u} \,,$$

$$\frac{\partial p}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} p = -\gamma p \boldsymbol{\nabla} \cdot \boldsymbol{u} \,,$$

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \right) = -\rho \boldsymbol{\nabla} \Phi - \boldsymbol{\nabla} p + \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} \,,$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) \,,$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \,,$$

$$\nabla^2 \Phi = 4\pi G \rho \,.$$

1

(a) In magnetohydrodynamics, the *cross-helicity* contained in a fixed volume V is defined as

$$H_\mathrm{c} = \int_V \boldsymbol{u} \cdot \boldsymbol{B} \, \mathrm{d}V \, .$$

Starting from the equations of ideal MHD, derive the conservation law

$$\frac{\partial}{\partial t}(\boldsymbol{u} \cdot \boldsymbol{B}) + \boldsymbol{\nabla} \cdot \left[\boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{B}) + (\tfrac{1}{2}u^2 + \Phi + h)\boldsymbol{B}\right] = T\boldsymbol{B} \cdot \boldsymbol{\nabla} s \, ,$$

where $h = e + p/\rho$. Name the quantities h, T and s appearing in this equation. Under what conditions is H_c independent of time?

[You may assume the identity $\boldsymbol{\nabla} \cdot (\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) - \boldsymbol{F} \cdot (\boldsymbol{\nabla} \times \boldsymbol{G})$.]

(b) The *Elsässer variables* of MHD are defined by

$$\boldsymbol{z}_\pm = \boldsymbol{u} \pm \boldsymbol{v}_\mathrm{a} \, ,$$

where $\boldsymbol{u}$ is the fluid velocity and $\boldsymbol{v}_\mathrm{a} = (\mu_0\rho)^{-1/2}\boldsymbol{B}$ is the Alfvén velocity. Starting from the equations of ideal MHD for an incompressible fluid of uniform density, show that the Elsässer variables satisfy

$$\frac{\partial \boldsymbol{z}_+}{\partial t} + \boldsymbol{z}_- \cdot \boldsymbol{\nabla}\boldsymbol{z}_+ = -\boldsymbol{\nabla}\psi \, , \qquad \frac{\partial \boldsymbol{z}_-}{\partial t} + \boldsymbol{z}_+ \cdot \boldsymbol{\nabla}\boldsymbol{z}_- = -\boldsymbol{\nabla}\psi \, , \qquad \boldsymbol{\nabla} \cdot \boldsymbol{z}_\pm = 0 \, ,$$

where ψ should be defined.

Deduce that the integrals

$$I_+ = \int_V |\boldsymbol{z}_+|^2 \, \mathrm{d}V \, , \qquad I_- = \int_V |\boldsymbol{z}_-|^2 \, \mathrm{d}V$$

are independent of time, if $\boldsymbol{u}$ and $\boldsymbol{B}$ satisfy suitable conditions (which should be stated) on the boundary of a fixed volume V.

Determine how these integrals are related to the (kinetic + magnetic) energy and the cross-helicity contained in the volume V.

2

A perfect gas of adiabatic exponent $\gamma = 3/2$ undergoes a steady, spherically symmetric flow in the exterior of a spherical mass M. Self-gravity, rotation and magnetic fields may be neglected.

Show that the problem can be reduced to the algebraic equation

$$\lambda \left(y^4 + 4y^{-1} \right) = x^4 + 4x^{-1} \, ,$$

where the dimensionless variables x and y are related to the Mach number $\mathcal{M}$ of the flow and the radial coordinate r by

$$x = \left(\frac{r}{r_{\mathrm{s}}} \right)^{1/5} , \qquad y = \mathcal{M}^{2/5} \, .$$

Relate the two constants λ and r_{s} to the physical parameters of the problem.

Sketch the solution curves in the (x, y) plane for various positive values of λ. Argue that the flow can undergo a sonic transition at $r = r_{\mathrm{s}}$, if λ has a special value. Find the simple analytical form of one of the two transonic solutions, and determine how the density, pressure and velocity depend on r in this special solution. Derive, and interpret physically, the approximate behaviour of the other transonic solution in the limits of large and small r.

If a steady transonic flow is arrested in an adiabatic shock after it has become supersonic, explain briefly why λ increases on passing through the shock, while the Bernoulli constant is unchanged. Deduce that the flow jumps to a subsonic solution branch, and sketch this on your graph.

3

Consider a plane-parallel model of a stellar atmosphere with uniform gravity $\boldsymbol{g} = -g\,\boldsymbol{e}_z$, where (x, y, z) are Cartesian coordinates. The basic state is in magnetostatic equilibrium, with density $\rho(z)$, pressure $p(z)$ and magnetic field $\boldsymbol{B} = B(z)\,\boldsymbol{e}_y$. Given that the isothermal sound speed c_s and the Alfvén speed v_a are independent of z in the basic state, determine the vertical profiles of density, pressure and magnetic field. Show that $\rho \propto \exp(-z/H)$, and relate the density scaleheight H to g, c_s and v_a.

You may assume that the linearized equation of motion governing a small displacement $\boldsymbol{\xi}$ to a magnetostatic equilibrium, neglecting self-gravity, for a perfect gas of adiabatic exponent γ, is

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = \boldsymbol{g}\,\delta\rho - \boldsymbol{\nabla}\delta\Pi + \frac{1}{\mu_0}(\delta\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B} + \boldsymbol{B}\cdot\boldsymbol{\nabla}\delta\boldsymbol{B}),$$

where the Eulerian perturbations of density, magnetic field and total pressure are given by

$$\begin{aligned}
\delta\rho &= -\rho\boldsymbol{\nabla}\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\boldsymbol{\nabla}\rho\,,\\
\delta\boldsymbol{B} &= \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{\xi} - \boldsymbol{B}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) - \boldsymbol{\xi}\cdot\boldsymbol{\nabla}\boldsymbol{B}\,,\\
\delta\Pi &= -\left(\gamma p + \frac{B^2}{\mu_0}\right)\boldsymbol{\nabla}\cdot\boldsymbol{\xi} + \frac{1}{\mu_0}\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{\xi}) - \boldsymbol{\xi}\cdot\boldsymbol{\nabla}\Pi\,.
\end{aligned}$$

Verify that these equations admit solutions in which

$$\begin{aligned}
\boldsymbol{\xi} &= \mathrm{Re}\left[\tilde{\boldsymbol{\xi}}\,\exp\left(\mathrm{i}k_x x + \mathrm{i}k_y y + \mathrm{i}k_z z + \frac{z}{2H} - \mathrm{i}\omega t\right)\right]\,,\\
\frac{\delta\Pi}{\rho} &= \mathrm{Re}\left[\psi\,\exp\left(\mathrm{i}k_x x + \mathrm{i}k_y y + \mathrm{i}k_z z + \frac{z}{2H} - \mathrm{i}\omega t\right)\right]\,,
\end{aligned}$$

where $\tilde{\boldsymbol{\xi}}$ and ψ are complex constants, $\boldsymbol{k}$ is a real wavevector with three components and ω is a complex frequency. Deduce the linearized equations in the algebraic form

$$\begin{aligned}
-\omega^2\tilde{\xi}_x &= -\mathrm{i}k_x\,\psi - k_y^2 v_\mathrm{a}^2\tilde{\xi}_x\,,\\
-\omega^2\tilde{\xi}_y &= -\mathrm{i}k_y\,(\psi + v_\mathrm{a}^2\Delta) - k_y^2 v_\mathrm{a}^2\tilde{\xi}_y\,,\\
-\omega^2\tilde{\xi}_z &= g\left(\Delta - \frac{\tilde{\xi}_z}{H}\right) - \left(\mathrm{i}k_z - \frac{1}{2H}\right)\psi - k_y^2 v_\mathrm{a}^2\tilde{\xi}_z\,,
\end{aligned}$$

with

$$\begin{aligned}
\psi &= -(v_\mathrm{s}^2 + v_\mathrm{a}^2)\Delta + v_\mathrm{a}^2\mathrm{i}k_y\tilde{\xi}_y + g\tilde{\xi}_z\,,\\
\Delta &= \mathrm{i}k_x\tilde{\xi}_x + \mathrm{i}k_y\tilde{\xi}_y + \left(\mathrm{i}k_z + \frac{1}{2H}\right)\tilde{\xi}_z\,,
\end{aligned}$$

where v_s is the adiabatic sound speed.

On what general grounds should we expect the roots of the dispersion relation to have real values of ω^2? Show that disturbances with $\omega^2 = 0$ and $k_y \neq 0$ satisfy

$$v_\mathrm{s}^2\Delta = g\tilde{\xi}_z\,,$$

$$-(k_x^2 + k_y^2)\psi = k_y^2 v_\mathrm{a}^2\left(\mathrm{i}k_z + \frac{1}{2H}\right)\tilde{\xi}_z\,.$$

Deduce that the condition for marginal stability is

$$\left(|\boldsymbol{k}|^2 + \frac{1}{4H^2}\right) k_y^2 v_{\mathrm{a}}^2 + \frac{g}{H}\left(1 - \frac{gH}{v_{\mathrm{s}}^2}\right)(k_x^2 + k_y^2) = 0\,.$$

Assuming that the boundary conditions do not constrain the admissible values of $\boldsymbol{k}$, deduce that the atmosphere is unstable if and only if $gH > v_{\mathrm{s}}^2$. Show that this condition is equivalent to

$$\beta(\gamma - 1) < 1\,,$$

where β is the ratio of gas pressure to magnetic pressure. Discuss briefly why the atmosphere becomes unstable if it is sufficiently strongly magnetized.

4

(a) A simplified model of a star or giant planet consists of a self-gravitating, incompressible fluid of mass M_1 and uniform density ρ_1. Solve for the hydrostatic equilibrium of such a body in the absence of external masses, neglecting rotation.

Explain why small oscillation modes of this system have irrotational displacements of the form

$$\boldsymbol{\xi} = \mathrm{Re}\left[\boldsymbol{\nabla} U(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega t}\right] ,$$

where $U(\boldsymbol{x})$ is a scalar potential satisfying Laplace's equation. Show that, when the effects of self-gravity are included, the mode frequencies are given by $\omega^2 = \omega_l^2$, where

$$\omega_l^2 = \frac{2l(l-1)}{2l+1}\frac{GM_1}{R_1^3} ,$$

R_1 is the radius of the body and l is a positive integer. Explain physically why $\omega^2 = 0$ in the case $l = 1$.

[You may assume that the interior and exterior solid spherical harmonics, $r^l Y_l^m$ and $r^{-l-1} Y_l^m$, are solutions of Laplace's equation, where l is a non-negative integer, m is an integer satisfying $|m| \leqslant l$ and $Y_l^m(\theta,\phi)$ is a spherical harmonic function. You may also assume that the Lagrangian pressure perturbation vanishes at the surface of the body.]

(b) The body is now placed in a circular orbit of radius a with a companion of mass M_2. The orbital frequency ω_o is given by

$$\omega_\mathrm{o}^2 = \frac{G(M_1+M_2)}{a^3} .$$

The dominant component of the tidal force per unit mass experienced by the first body is of the form

$$\mathrm{Re}\left[-\boldsymbol{\nabla}\Psi(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega t}\right] ,$$

with

$$\Psi(\boldsymbol{x}) = \psi\frac{GM_2}{a^3} r^l Y_l^m ,$$

where $l = 2$, $\omega = 2\omega_\mathrm{o}$ and ψ is a dimensionless constant of order unity. Explain briefly why the tidal forcing has this form. [Spin may be neglected, and you are not required to determine the value of ψ.]

By modifying the linearized equation of motion used in part (a) to include the tidal force, and assuming that the displacement has the same frequency as the tidal force, show that the radial displacement at the surface of the first body is

$$\epsilon R_1\left(\frac{\omega_2^2}{\omega^2-\omega_2^2}\right)\mathrm{Re}\left(\psi Y_l^m\,\mathrm{e}^{-\mathrm{i}\omega t}\right) ,$$

according to linear theory, where

$$\epsilon = \frac{5}{2}\frac{M_2}{M_1}\left(\frac{R_1}{a}\right)^3$$

and ω_2 is the frequency of the $l = 2$ f mode determined in part (a). Deduce that the tidal forcing resonates with this mode when

$$\left(\frac{R_1}{a}\right)^3 = \frac{1}{5}(1 - 2\epsilon)\,.$$

END OF PAPER