UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 7 June, 2017 9:00 am to 12:00 pm

PAPER 312

ADVANCED COSMOLOGY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider the 3+1 formalism for general relativity with line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + {}^{(3)}g_{ij}(dx^i - N^i dt)(dx^j - N^j dt)\,, \qquad (*)$$

where $N(x^i,t)$ is the lapse function, $N^i(x^i,t)$ is the shift vector, and ${}^{(3)}g_{ij}(x^i,t)$ is the three-metric on constant time spacelike hypersurfaces Σ. (Latin indices $i = 1,2,3$; Greek $\mu = 0,1,2,3\,$.) We can define quantities on the three-dimensional hypersurface Σ using the projection tensor

$$P^\mu{}_\alpha = \delta^\mu{}_\alpha + n^\mu n_\alpha\,,$$

where $n_\alpha = (-N,0,0,0)$ is a future-pointing normal to Σ with $n_\mu n^\mu = -1$.

(i) Verify that $P^\mu{}_\alpha\, n_\mu = 0$. Show that the projected three-dimensional covariant derivative of the three-metric ${}^{(3)}g_{ij}$ on Σ vanishes, that is,

$${}^{(3)}g_{ij|k} \equiv P^\alpha{}_i P^\beta{}_j P^\ell{}_k \nabla_\ell\,(P^\mu{}_\alpha P^\nu{}_\beta\, g_{\mu\nu}) = 0\,,$$

where ∇_ℓ is the four-dimensional covariant derivative.

(ii) We can define the extrinsic curvature $K_{\mu\nu}$ for the constant time slice Σ as $K_{\mu\nu} = -P^\alpha{}_\mu P^\beta{}_\nu\,\nabla_\alpha n_\beta\,$. Show that $n^\mu \nabla_\nu n_\mu = 0$ and hence that $K_{\mu\nu} = -P^\alpha{}_\mu \nabla_\alpha n_\nu\,$. The Frobenius theorem states that, if n^μ is hypersurface orthogonal, then it satisfies $n_{[\alpha}\nabla_\beta n_{\lambda]} = 0$. (Here, the permutation symbol is $[\alpha\beta\lambda] = \alpha\beta\lambda + \beta\lambda\alpha + \lambda\alpha\beta - \beta\alpha\lambda - \alpha\lambda\beta - \lambda\beta\alpha$.) Use this result to show that $K_{\mu\nu}$ is a symmetric tensor.

(b) For a flat background FLRW model, we can linearise the 3+1 metric $(*)$ for scalar perturbations using the substitutions

$$N = \bar{N}(1+\Psi)\,, \qquad N_i = -a^2 B_{,i}\,, \qquad {}^{(3)}g_{ij} = a^2[(1-2\Phi)\delta_{ij} + 2E_{,ij}]\,, \qquad (\dagger)$$

where $\Psi(\mathbf{x},t), \Phi(\mathbf{x},t), B(\mathbf{x},t)$ and $E(\mathbf{x},t)$ are arbitrary functions, while the scale factor $a(t)$ and the background lapse $\bar{N}(t)$ depend only on time.

(i) Given that the extrinsic curvature is $K_{ij} = -\frac{1}{2N}\left({}^{(3)}g_{ij,0} + N_{i|j} + N_{j|i}\right)$, show that the linearised curvature for the perturbed metric $(\dagger)$ becomes

$$K^i{}_j = {}^{(3)}g^{ik}K_{kj} = -H\delta^i_j + \tfrac{1}{3}\kappa\delta^i_j - (\partial^i\partial_j - \tfrac{1}{3}\triangle\delta^i_j)\chi\,,$$

where $\kappa \equiv 3(\dot{\Phi}/\bar{N} + H\Psi) - \triangle\chi$ and $\chi \equiv -(a^2/\bar{N})(B - \dot{E})$, with $H = \dot{a}/(\bar{N}a)$ and the Laplacian $\triangle = \partial_i\partial^i = \nabla^2/a^2$.

(ii) Linearise the Einstein equation ${}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi G\rho$ to find the corresponding perturbation equation in terms of the variables κ, Φ and $\delta\rho$, where the perturbed energy density $\rho(\mathbf{x},t) \equiv \bar{\rho}(t) + \delta\rho(\mathbf{x},t)$.

[*You may assume the linearised Ricci scalar is* ${}^{(3)}R = 4\triangle\Phi$.]

2

(i) Explain briefly why the stress–energy tensor of a gas of photons, described by a one-particle distribution function f, is given by

$$T^{\mu\nu} = \int \frac{d^3\mathbf{p}}{E(\mathbf{p})} f p^\mu p^\nu \,,$$

where $\mathbf{p}$ is the 3-momentum and $E(\mathbf{p}) = |\mathbf{p}|$ is the energy of a photon defined with respect to an orthonormal tetrad, and p^μ is the photon 4-momentum.

In linear perturbation theory, the distribution function can be written as

$$f(\eta, \mathbf{x}, \epsilon, \mathbf{e}) = \bar{f}(\epsilon)\left[1 - \frac{d\ln \bar{f}}{d\ln \epsilon}\Theta(\eta, \mathbf{x}, \mathbf{e})\right] ,$$

where $\bar{f}(\epsilon)$ is the unperturbed distribution function, which depends only on comoving energy $\epsilon = aE$ with a the scale-factor. Here, $\mathbf{e}$ is the photon direction (with $\mathbf{p} = E\mathbf{e}$), η is conformal time, $\mathbf{x}$ is comoving position, and Θ describes the dimensionless temperature anisotropies. Denoting the energy density relative to the orthonormal tetrad as $\bar{\rho}(1+\delta)$, where $\bar{\rho}$ is the unperturbed energy density, the pressure as $\bar{P}+\delta P$, with $\bar{P}$ the unperturbed pressure, the momentum density as $\mathbf{q} = (\bar{\rho}+\bar{P})\mathbf{v}$, and the anisotropic stress as $\Pi^{\hat{\imath}\hat{\jmath}}$, show that

$$\bar{\rho} = \frac{4\pi}{a^4}\int d\epsilon\, \epsilon^3 \bar{f}(\epsilon) \qquad \text{and} \qquad \bar{P} = \frac{1}{3}\bar{\rho} \,.$$

Show further that

$$\delta = 4\int \frac{d\mathbf{e}}{4\pi}\,\Theta \,, \qquad \mathbf{v} = 3\int \frac{d\mathbf{e}}{4\pi}\,\Theta\mathbf{e} \,, \qquad \Pi^{\hat{\imath}\hat{\jmath}} = -4\bar{\rho}\int \frac{d\mathbf{e}}{4\pi}\,\Theta\left(e^{\hat{\imath}}e^{\hat{\jmath}} - \frac{1}{3}\delta^{\hat{\imath}\hat{\jmath}}\right) ,$$

and $\delta P = \bar{\rho}\delta/3$.

(ii) For freely-propagating photons, the distribution function is conserved along the photon path in phase space. Show that in linear perturbation theory

$$\frac{\partial \Theta}{\partial \eta} + \mathbf{e}\cdot\boldsymbol{\nabla}\Theta - \frac{d\ln\epsilon}{d\eta} = 0 \,. \qquad (*)$$

For scalar perturbations in the conformal Newtonian gauge, $d\ln\epsilon/d\eta = \dot{\phi} - \mathbf{e}\cdot\boldsymbol{\nabla}\psi$, where ϕ and ψ are the metric potentials and overdots denote partial differentiation with respect to η. By integrating $(*)$ over $\mathbf{e}$, after multiplication by suitable tensor products of $\mathbf{e}$, derive the continuity equation

$$\dot{\delta} + \frac{4}{3}\boldsymbol{\nabla}\cdot\mathbf{v} - 4\dot{\phi} = 0$$

and the Euler equation

$$\dot{v}^{\hat{\imath}} + \frac{1}{4}\delta^{ij}\partial_j\delta - \frac{3}{4\bar{\rho}}\partial_j\Pi^{\hat{\imath}\hat{\jmath}} + \delta^{ij}\partial_j\psi = 0 \,.$$

[*You may wish to use* $\int d\mathbf{e}\left(e^{\hat{\imath}}e^{\hat{\jmath}} - \delta^{\hat{\imath}\hat{\jmath}}/3\right) = 0$.]

(iii) What is the physical interpretation of the 4-vector

$$J^\mu = \int \frac{d^3\mathbf{p}}{E(\mathbf{p})} f p^\mu \,?$$

Discuss under what circumstances $\nabla_\mu J^\mu = 0$.

3

Vector perturbations of a spatially-flat universe can be described by the line element

$$ds^2 = a^2(\eta)\left[-d\eta^2 + 2B_i d\eta dx^i + \delta_{ij} dx^i dx^j\right] ,$$

where $a(\eta)$ is the scale factor at conformal time η and B_i is a divergence-free 3-vector ($\delta^{ij}\partial_i B_j = 0$). Throughout this question you should work to first order in the perturbation B_i.

(i) Show that the tetrad vectors with coordinate components

$$(E_0)^\mu = a^{-1}\delta_0^\mu \qquad \text{and} \qquad (E_i)^\mu = a^{-1}\left(B_i\delta_0^\mu + \delta_i^\mu\right)$$

are orthonormal.

By writing the photon 4-momentum as $p^\mu = a^{-1}\epsilon\left[(E_0)^\mu + e^{\hat{\imath}}(E_i)^\mu)\right]$, where ϵ is the comoving energy and $e^{\hat{\imath}}$ are the direction cosines, show that

$$\frac{1}{\epsilon}\frac{d\epsilon}{d\eta} + e^{\hat{\imath}}\dot{B}_i = 0\,,$$

where overdots denote partial differentiation with respect to conformal time.

[*You may assume the following connection coefficients:*

$$\Gamma^0_{00} = \mathcal{H}\,, \qquad \Gamma^0_{0i} = \mathcal{H}B_i\,, \qquad \Gamma^0_{ij} = \mathcal{H}\delta_{ij} - \frac{1}{2}\left(\partial_i B_j + \partial_j B_i\right)\,,$$

where $\mathcal{H} = \dot{a}/a$.]

(ii) Consider a single Fourier mode of the metric perturbation B_i, with wavevector $\mathbf{k} = k\hat{\mathbf{z}}$ lying along the z-axis. Explain briefly why this can be written as a superposition of two helicity states, labelled $\pm$, each of the form

$$B_i^{(\pm)}(\eta, k\hat{\mathbf{z}}) = \frac{i}{\sqrt{2}}B^{(\pm)}(\eta, k\hat{\mathbf{z}})m_i^{(\pm)}(\hat{\mathbf{z}})\,, \qquad (*)$$

where the complex basis vectors $m_i^{(\pm)}(\hat{\mathbf{z}}) = (\delta_{i1} \pm i\delta_{i2})/\sqrt{2}$.

The Boltzmann equation for the fractional temperature anisotropy in the cosmic microwave background, $\Theta(\eta, \mathbf{x}, \mathbf{e})$, can be written for vector perturbations as

$$\frac{\partial\Theta}{\partial\eta} + \mathbf{e}\cdot\boldsymbol{\nabla}\Theta + \mathbf{e}\cdot\dot{\mathbf{B}} = \dot{\tau}\Theta - \frac{1}{10}\dot{\tau}\sum_{m=-2}^{2}\Theta_{2m}Y_{2m}(\mathbf{e}) - \dot{\tau}\mathbf{e}\cdot\mathbf{v}_b\,,$$

where $\mathbf{v}_b$ is the baryon peculiar velocity, τ is the optical depth to Thomson scattering, and Θ_{lm} are the spherical multipoles of Θ. For the $\pm$ helicity states given in $(*)$, show that the source terms $\mathbf{e}\cdot\mathbf{B}$ and $\mathbf{e}\cdot\mathbf{v}_b$ in the Boltzmann equation generate anisotropies with $m = \pm 1$ only, so that we can write

$$\Theta^{(\pm)}(\eta, k\hat{\mathbf{z}}, \mathbf{e}) = \frac{1}{\sqrt{2}}\sum_{l\geqslant 1}(-i)^l\sqrt{\frac{2l+1}{4\pi}}\Theta_l^{(\pm)}(\eta, k\hat{\mathbf{z}})Y_{l\,\pm 1}(\mathbf{e})\,.$$

Show that the $\Theta_l^{(\pm)}(\eta, k\hat{\mathbf{z}})$ satisfy the Boltzmann hierarchy

$$\dot{\Theta}_l^{(\pm)} + k\left[\frac{\sqrt{(l+1)^2-1}}{2l+3}\Theta_{l+1}^{(\pm)} - \frac{\sqrt{l^2-1}}{2l-1}\Theta_{l-1}^{(\pm)}\right] = \dot{\tau}\Theta_l^{(\pm)} \\ - \frac{1}{10}\dot{\tau}\Theta_2^{(\pm)}\delta_{l2} \mp \left(\dot{B}^{(\pm)} + \dot{\tau}v_b^{(\pm)}\right)\delta_{l1}\,,$$

where $v_b^{(\pm)}(\eta, k\hat{\mathbf{z}})$ is the amplitude of the $\pm$ helicity state of $\mathbf{v}_b(\eta, k\hat{\mathbf{z}})$, defined by analogy to $(*)$.

[*Note that* $Y_{1\,\pm1} = \mp\sqrt{3/(8\pi)}\sin\theta e^{\pm i\phi}$, *and*

$$\cos\theta\, Y_{lm} = \sqrt{\frac{(l+1)^2-m^2}{(2l+3)(2l+1)}}Y_{l+1\,m} + \sqrt{\frac{l^2-m^2}{(2l+1)(2l-1)}}Y_{l-1\,m}\,.$$]

4

(a) According to the in-in formalism, the leading order correction to an operator Q during inflation is given by the expectation value

$$\langle Q(t)\rangle = \mathcal{R}e\left\langle -2iQ^I(t)\int_{-\infty(1-i\mathcal{E})}^{t} H^I_{\rm int}(t')dt'\right\rangle\,, \qquad (\dagger)$$

where $H^I_{\rm int}$ is the interaction Hamiltonian for single-field inflation at third-order. We assume that Q and $H^I_{\rm int}$ are given in terms of the linear density perturbation ζ, which is a Gaussian random field with power spectrum,

$$\langle\zeta(\mathbf{k},\tau)\zeta(\mathbf{k}',\tau)\rangle = (2\pi)^3 u_{\mathbf{k}}(\tau)\,u^*_{\mathbf{k}}(\tau)\,\delta(\mathbf{k}+\mathbf{k}')\,,$$

with mode functions $u_{\mathbf{k}}(\tau)$ and their conformal time derivatives $u'_{\mathbf{k}}(\tau)$ given by

$$u_{\mathbf{k}}(\tau) = \frac{H}{\sqrt{4\epsilon M_{\rm Pl}^2 c_{\rm s} k^3}}\,(1+ikc_{\rm s}\tau)\,e^{-ikc_{\rm s}\tau}\,,\quad u'_{\mathbf{k}}(\tau) = \frac{H}{\sqrt{4\epsilon M_{\rm Pl}^2 c_{\rm s} k^3}} c_{\rm s}^2 k^2\tau e^{-ikc_{\rm s}\tau}\,,$$

where we have allowed for a non-trivial sound speed $c_{\rm s}$ different from unity. During inflation we assume the scale factor is $a \approx -1/(H\tau)$ with conformal time τ (i.e. $dt = ad\tau$) and $\dot\zeta = d\zeta/dt$ and $\zeta' = d\zeta/d\tau$) and that H, ϵ and $c_{\rm s}$ are effectively constant.

(i) For the interaction Hamiltonian,

$$H^I_{\rm int} = -M_{\rm Pl}^2\int d^3x\,\frac{a^3\epsilon}{H}\frac{(\epsilon+1-c_{\rm s}^2)}{c_{\rm s}^2}\,\dot\zeta^3\,,$$

show that the three point correlator (bispectrum) of ζ reduces to the following:

$$\begin{aligned}\langle\zeta(\mathbf{k}_1,0)\zeta(\mathbf{k}_2,0)\zeta(\mathbf{k}_3,0)\rangle = 6\mathcal{R}e\Bigg(&-2i\int d\tau\; M_{\rm Pl}^2\frac{\epsilon(\epsilon+1-c_{\rm s}^2)}{c_{\rm s}^2H^2\tau} \\ &\times\; u_{\mathbf{k}_1}(0)\,u_{\mathbf{k}_2}(0)\,u_{\mathbf{k}_3}(0)\,u'^*_{\mathbf{k}_1}(\tau)\,u'^*_{\mathbf{k}_2}(\tau)\,u'^*_{\mathbf{k}_3}(\tau)\,(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\Bigg)\,,\end{aligned} \qquad (\ddagger)$$

stating clearly any assumptions that you have made.

(ii) Substitute the mode functions explicitly in the expression (‡) and integrate to find an expression for the bispectrum $B(k_1,k_2,k_3)$. Briefly discuss the level of non-Gaussianity in the limits $c_{\rm s} \ll 0$ and $c_{\rm s}\to 1$ in comparison to standard single field inflation. Discuss which triangle configurations (if any) dominate the bispectrum signal-to-noise by considering the behaviour of the shape function $S(k_1,k_2,k_3) \sim (k_1k_2k_3)^2B(k_1,k_2,k_3)$.

(b) The reduced CMB bispectrum $b_{l_1l_2l_3}$ arises from the primordial bispectrum $B(k_1,k_2,k_3)$ through the integral expression:

$$\begin{aligned} b^{\rm loc}_{l_1l_2l_3} = \left(\frac{2}{\pi}\right)^3\int dx\int & dk_1\,dk_2\,dk_3\,(x\,k_1k_2k_3)^2B(k_1,k_2,k_3) \\ &\times\;\Delta_{l_1}(k_1)\Delta_{l_2}(k_2)\Delta_{l_3}(k_3)\,j_{l_1}(k_1x)j_{l_2}(k_2x)j_{l_3}(k_3x)\,.\end{aligned} \qquad (*)$$

Assume that we have a 'constant' primordial bispectrum given by

$$B(k_1, k_2, k_3) = f_{\rm NL}[P(k_1)P(k_2)P(k_3)]^{2/3} \approx f_{\rm NL}\, 4\pi^4 \Delta_\zeta^4/(k_1 k_2 k_3)^2\,,$$

for which we ignore the scale-dependence of the variance Δ_ζ^2. Restrict attention to large angular scales ($l \ll 200$), where we can approximate the transfer functions by $\Delta_l(k) = \frac{1}{5} j_l(\Delta\tau k)$ with $\Delta\tau = \tau_0 - \tau_{\rm dec}$, and integrate the reduced bispectrum (*) to find

$$b_{l_1 l_2 l_3} \;=\; \frac{\mathcal{A} f_{\rm NL}}{(2l_1+1)(2l_2+1)(2l_3+1)} \left[\frac{1}{l_1+l_2+l_3+3} + \frac{1}{l_1+l_2+l_3}\right].$$

Here, you should state the magnitude of the constant $\mathcal{A}$. Briefly comment on the scale-dependence of this result.

[*Hint:* You may assume the following integral for Bessel function products:

$$\int dk\, j_l(k) j_l(kx) = \begin{cases} \pi\, x^{-(l+1)}/(2(2l+1)), & x > 1\,, \\ \pi\, x^l/(2(2l+1)), & x < 1\,. \end{cases}$$
]

END OF PAPER