UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 9 June, 2017 9:00 am to 12:00 pm

PAPER 311

BLACK HOLES

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) In an axisymmetric, asymptotically flat spacetime with axial Killing field m^a, we may define the total angular momentum, J, by

$$J = \frac{1}{16\pi} \int_S \star \mathrm{d}m \,,$$

where the integral is taken over a sphere, S, in the asymptotic region where the stress energy tensor T_{ab} is assumed to vanish.

(i) Show that J is independent of S and that J is given by

$$J = -\int_\Sigma \star J' \,,$$

where the hypersurface Σ is chosen so that m^a is tangent to Σ and $J'_a = T_{ab} m^b$.

(ii) Consider the emission of gravitational waves in an axisymmetric configuration. What is the physical interpretation of the fact that, for such configurations, J is independent of S?

(b) Suppose two widely separated Kerr black holes with parameters (M_1, J_1) and (M_2, J_2) are initially at rest in an axisymmetric configuration. Assume that these black holes fall together and coalesce into a single black hole.

(i) What is the total angular momentum of the final black hole?

(ii) Derive an upper limit for the energy radiated away in this process. Show that this upper limit depends on the relative sign of the spins of the two initial black holes. Discuss the physical significance of this dependence on the relative sign. *[You may assume that the area of the intersection of the future even horizon of a Kerr black hole, with mass M and angular momentum J, and a partial Cauchy surface is given by:*

$$A = 8\pi \left(M^2 + \sqrt{M^4 - J^2} \right) .]$$

2

(a) What is a null geodesic congruence? Define the expansion, rotation and shear of a null geodesic congruence.

(b) Derive Raychaudhuri's equation

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^{ab}\hat{\sigma}_{ab} + \hat{\omega}^{ab}\hat{\omega}_{ab} - R_{ab}U^a U^b\,,$$

for an affinely parametrised null congruence of geodesics with tangent vector U^a.

(c) Assume the spacetime satisfies the Einstein equation with matter obeying the null energy condition. Show that if $\theta = \theta_0 < 0$ at a point p on a generator γ of a null hypersurface, then $\theta \to -\infty$ within finite affine parameter distance $2/|\theta_0|$, provided γ extends this far. You may assume that if the congruence contains the generators of a null hypersurface $\mathcal{N}$, then $\hat{\omega}_{ab} = 0$ on $\mathcal{N}$.

(d) State and prove Penrose's singularity theorem. You may assume that if S is a two-dimensional orientable spacelike submanifold of a globally hyperbolic spacetime, then every $p \in \dot{J}^+(S)$ lies on a future-directed null geodesic starting from S which is orthogonal to S and has no points conjugate to S between S and p. You may also assume that $\dot{J}^+(S)$ is an achronal submanifold and that p is conjugate to S if $\theta \to -\infty$ at p.

3

A five-dimensional black hole spacetime has metric

$$\mathrm{d}s^2 = -\frac{f(r)}{h(r)}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left[h(r)\left(\mathrm{d}\psi + \frac{\cos\theta}{2}\mathrm{d}\phi - \Omega(r)\,\mathrm{d}t\right)^2 + \frac{1}{4}\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]$$

where

$$f(r) = 1 - \frac{r_0^2}{r^2} + \frac{\alpha^2\,r_0^2}{r^4} \equiv \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4}\,, \quad h(r) = 1 + \frac{\alpha^2\,r_0^2}{r^4}\,, \quad \Omega(r) = \frac{\alpha\,r_0^2}{r^4\,h(r)}\,,$$

and

$$0 \leqslant \psi, \phi \leqslant 2\pi\,, \quad 0 \leqslant \theta \leqslant \pi\,, \quad \text{and} \quad r_\pm^2 = \frac{r_0^2}{2} \pm \frac{r_0^2}{2}\sqrt{1 - \frac{4\alpha^2}{r_0^2}}\,.$$

You may assume that $\alpha > 0$, $\alpha/r_0 \leqslant 1/2$ and

$$R^{abcd}R_{abcd} = \frac{24r_0^4\left(16\,\alpha^4 - 16\,\alpha^2 r^2 + 3\,r^4\right)}{r^{12}}\,.$$

(a) Calculate the Komar mass M of this solution defined by

$$M = \frac{3}{32\pi}\lim_{r\to+\infty}\int \star\,\mathrm{d}k\,,$$

where $k^a = (\partial/\partial t)^a$. The integral is taken over a constant $t\,,r$ surface at infinity and the orientation is $-\mathrm{d}t\wedge\mathrm{d}r\wedge\mathrm{d}\psi\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi$.

(b) Show that the submanifold of the spacetime corresponding to $\theta = \pi/2$, $\phi = const.$ is totally geodesic, *i.e.* a geodesic *initially* tangent to it, will remain tangent. Furthermore, show that geodesics with zero angular momenta satisfy

$$\frac{\mathrm{d}\psi}{\mathrm{d}\lambda} = \Omega(r(\lambda))\,\frac{\mathrm{d}t}{\mathrm{d}\lambda}\,.$$

(c) Show that one can define a quantity $r_\star$ such that $u = t - r_\star$ and $v = t + r_\star$ are constant on the outgoing and ingoing null geodesics of (b), respectively. (You may express $r_\star$ as an integral.)

(d) Define the *black hole region* of an asymptotically flat spacetime. Prove that the region $r_- < r < r_+$ is within the black hole region. Sketch the Penrose diagram(s) for the submanifold defined in (b).

4

Write an essay on the relation of black holes to thermodynamics.

You should start with a statement of the laws of black hole mechanics and explain why they are analogous to the laws of thermodynamics. In particular, by consideration of the first law, you should explain how the connection between surface gravity and temperature leads to the Bekenstein-Hawking formula for the entropy of a black hole.

Next, you should consider a quantum scalar field Φ satisfying the wave equation $\nabla_\mu \nabla^\mu \Phi = 0$ in a globally hyperbolic non-stationary spacetime that is asymptotic to Minkowski spacetime in the far past and far future, and explain how the vacuum state can evolve to a non-vacuum state. You should then explain briefly how your results apply to late-time Hawking radiation from a Schwarzschild black hole formed from gravitational collapse.

You should conclude with a brief discussion of some of the implications of Hawking radiation for black holes.

END OF PAPER