UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 7 June, 2017 1:30 pm to 4:30 pm

PAPER 310

COSMOLOGY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) The Friedmann and acceleration equations describing our universe are,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right) .$$

Use these to show that,

$$\frac{d\Omega}{d\ln a} = (1+3\omega)\,\Omega\,(\Omega - 1) \ ,$$

where you should define both ω and Ω.

(b) Perturb Ω around $\Omega = 1$ and show that the perturbation grows as $a^{1+3\omega}$. Use this to describe the **flatness problem**.

(c) Another key problem of the standard big bang model is the **horizon problem**, which we summarised in lectures as that for ordinary matter the Hubble radius is always growing,

$$\frac{d}{dt}(\mathcal{H}^{-1}) > 0\,.$$

By showing that,

$$\frac{d}{dt}\left(\mathcal{H}^{-1}\right) = A(1+3\omega)\,,$$

where A is always positive, argue that any universe that has a flatness problem will generally always have a horizon problem and vice versa.

(d) During phase transitions in the early universe topological defects can be produced. One interesting example are cosmic strings, which are one dimensional defects. Consider a universe that contains a network of infinite non-interacting cosmic strings. The energy density of such a network only scales with a via dilution, so $\rho \propto 1/a^2$.

Use the scaling of the energy density with,

$$\dot{\rho} = -3H(1+\omega)\rho\,,$$

to find the effective equation of state, ω, for a string network. Briefly comment on the horizon and flatness problems in this universe.

2

(a) The continuity and Euler equations for matter perturbations are

$$\begin{aligned}\delta_m' &= -\nabla \cdot \mathbf{v}_m + 3\Phi' \\ \mathbf{v}_m' &= -\mathcal{H}\mathbf{v}_m - \nabla\Phi \,.\end{aligned}$$

Use these to show that

$$\delta_m'' + \mathcal{H}\delta_m' = \nabla^2\Phi + 3\left(\Phi'' + \mathcal{H}\Phi'\right) \,. \tag{*}$$

(b) Now take the Friedmann equation for a flat universe containing matter and radiation,

$$\mathcal{H}^2 = H^2\left(\Omega_m + \Omega_r\right)a^2 \,.$$

and show it can be re-written it as

$$\mathcal{H}^2 = \frac{H_0^2\Omega_{m,0}^2}{\Omega_{r,0}}\left(\frac{1}{y} + \frac{1}{y^2}\right) \,,$$

where you should define y.

(c) Argue why during the radiation era the contribution from radiation to the gravitational perturbation, Φ, can be neglected when considering matter on subhorizon scales.

(d) The argument in **(c)** allows us to replace the RHS of (*) with $\frac{3H_0^2\Omega_{m,0}^2}{2\Omega_{r,0}}\frac{1}{y}\delta_m$. Use this, with the result from part **(b)**, to derive the Mészáros equation

$$\frac{d^2\delta_m}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\delta_m}{dy} - \frac{3}{2y(1+y)}\delta_m = 0 \,.$$

(e) The solutions of the Mészáros equation show that matter perturbations grow as $\delta_m \propto a$ during the matter era but only as $\delta_m \propto \ln(a)$ during the radiation era. Briefly describe what effect this has on the matter power spectrum.

3

(a) Use the first law of thermodynamics to show that the entropy in a comoving volume is conserved for a gas of particles in kinetic equilibrium in an expanding universe (you may assume that the chemical potential vanishes).

Show that the entropy density is given by,

$$s = \frac{\rho + P}{T}\,.$$

[*Hint: You are free to use the result* $\frac{\partial P}{\partial T} = \frac{\rho+P}{T}$.]

(b) Consider a hot big bang universe at temperatures $T \gg 100$ GeV consisting of the Standard Model and, in addition, a relativistic 'dark' scalar field ϕ. For $T > T_d$ the Standard Model particles and ϕ are in kinetic equilibrium, but for $T < T_d$ the field ϕ is decoupled from the Standard Model. The field ϕ self-interacts through the processes $\phi\phi \leftrightarrow \phi\phi$ and $\phi\phi \leftrightarrow \phi\phi\phi$.

What is the chemical potential of ϕ? Find the ratio of the entropy densities before decoupling,

$$\xi(T > T_d) \equiv \frac{s_{SM}}{s_\phi}\,,$$

in terms of $g_\star^{\rm SM}$ and $g_\star^\phi$. Explain how it evolves with time.

c) Some time after decoupling ϕ will become non-relativistic. The Bose-Einstein number density is,

$$n = \frac{g}{2\pi^2} T^3 \int_0^\infty dy \frac{y^2}{e^{\sqrt{y^2+x^2}} - 1}$$

where $x = m_\phi/T_\phi$ and $y = p/T$. For $T \ll m_\phi$ show that

$$s_\phi \simeq \frac{m_\phi^3}{(2\pi)^{3/2}} x^{-1/2} e^{-x}\,.$$

Use this result, with the expression for the entropy density for relativistic ϕ,

$$s_\phi = \frac{2\pi^2}{45} g_{\star S}^\phi(T_\phi)\, T_\phi^3\,,$$

to show that

$$\frac{T_\gamma}{T_\phi} \simeq k \left(\frac{\xi}{g_\star^{\rm SM}}\right)^{1/3} x^{5/6} e^{-x/3}\,,$$

where k is a numerical factor to be determined.

[*Hint: You may find the following integral useful:* $\int_0^\infty dy y^p e^{-y^2} = \frac{1}{2}\Gamma\left(\frac{p+1}{2}\right)$.]

d) Does the temperature of the ϕ particles decrease faster or slower than the photon temperature? Can you provide a physical explanation for this behaviour?

4

(a) The Friedmann equations for a flat universe are:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_{pl}^2}\rho\,, \qquad \frac{\ddot{a}}{a} = -\frac{1}{6M_{pl}^2}\left(\rho + 3P\right)\,.$$

For a universe containing a single scalar field, for which

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \qquad P = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,,$$

derive the Klein-Gordon equation:

$$\ddot{\phi} + 3H\dot{\phi} = -V_{,\phi}$$

(b) The two slow-roll parameters are

$$\epsilon \equiv -\frac{d\ln H}{d\ln a}\,, \qquad \eta \equiv \frac{d\ln \epsilon}{d\ln a}\,.$$

Briefly describe what they parameterise and what is required for inflation to occur. Show that they are equivalent to,

$$\epsilon = -\frac{\dot{H}}{H^2}\,, \qquad \eta = \frac{\dot{\epsilon}}{H\epsilon}\,.$$

By making the slow-roll approximations, which you should state, derive the following:

$$H^2 \approx \frac{V}{3M_{pl}^2}\,, \quad 3H\dot{\phi} \approx -V_{,\phi}\,, \quad \epsilon \approx \frac{M_{pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \ (\equiv \epsilon_V)\ , \quad 2\epsilon - \frac{1}{2}\eta \approx M_{pl}^2\frac{V_{,\phi\phi}}{V}\ (\equiv \eta_V)\ .$$

(c) Consider a class of inflationary models described by $V = \lambda\phi^n$ where λ is a positive constant, $\phi > 0$, and $n \in \{2, 3, 4\}$. Show the slow roll conditions are only satisfied for

$$\left(\frac{\phi}{M_{pl}}\right)^2 \geqslant 2n$$

If we take the equality above as the end of inflation show that for 60 efolds we would need to have started at a field position

$$\left(\frac{\phi}{M_{pl}}\right)^2 = 122n$$

and use $n_s = 1 - 2\epsilon - \eta$ and $r = 16\epsilon$ to show that this leads to

$$n_s = 1 - \frac{n+2}{122}\,, \qquad r = \frac{4n}{61}\ (\approx 0.066n)$$

(d) For $n \in \{2, 3, 4\}$ consider the viability of these models with respect to the current constraints $0.956 < n_s < 0.980$ and $r < 0.07$.

(e) Are there any other obvious theoretical issues with these models?

END OF PAPER