UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 2 June, 2017 9:00 am to 12:00 pm

PAPER 309

GENERAL RELATIVITY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let A be a diagonal square matrix $A = \mathrm{diag}({a_1}^2, {a_2}^2, \ldots, {a_{n+1}}^2)$, where $a_i \neq 0$. Show, by constructing an atlas consisting of two charts, that the n–dimensional ellipsoid

$$\mathcal{Q}_n = \{\mathbf{x} \in \mathbb{R}^{n+1}, \mathbf{x}A\mathbf{x}^T = 1\}$$

is a smooth manifold.

(b) Let $\psi : \mathcal{Q} \to N$ be a smooth map between smooth manifolds. Define the push forward $\psi_*(X)$ of a vector field X on $\mathcal{Q}$.

(c) Let $(\rho > 0, \phi)$ be local coordinates on an open set $U \subset \mathcal{Q}$, where

$$\mathcal{Q} = \{(x, y, z) \in \mathbb{R}^3, x^2 + a^2(y^2 + z^2) = 1\}, \quad \text{where} \quad a \neq 0$$

such that $y + iz = \rho \exp(i\phi)$. Find the push–forward K of the vector field $\partial/\partial\phi$ from U to $\mathbb{R}^3$, and construct the integral curves of K in $\mathbb{R}^3$

(d) For the surface $\mathcal{Q}$ defined in (c), determine the metric g induced from the Euclidean metric $ds^2 = dx^2 + dy^2 + dz^2$ on $\mathbb{R}^3$. Hence find a non–zero function Ω on U such that $\Omega^2 g$ is flat on U.

2

(a) Let $g = g_{\mu\nu}dx^\mu dx^\nu$ be a Lorentzian metric on a manifold M, and $u \to x^\mu(u)$ be a timelike curve. Show that the Euler–Lagrange equations with Lagrangians $L = \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$, and $\mathcal{L} = (1/2)L^2$ share the same integral curves.

(b) Let $\phi : \mathbb{R}^3 \to \mathbb{R}$. Consider a family of Lorentzian metrics on $M = \mathbb{R}^3 \times \mathbb{R}$ parametrised by a non–zero constant c

$$g(c) = -c^2 e^{2\phi(x,y,z)/c^2} dt^2 + dx^2 + dy^2 + dz^2.$$

Compute the Christoffel symbols of the Levi–Civita connection $\nabla^{(c)}$ of $g(c)$. Comment on the limit $c \to \infty$, and find the Christoffel symbols of the connection $\nabla^{(\infty)}$.

(c) Find the Ricci tensor of $\nabla^{(\infty)}$, and show that it vanishes if and only if the function ϕ satisfies the Laplace equation on $\mathbb{R}^3$.

(d) Let $\mathbf{E}$ and $\mathbf{B}$ be vector fields on $\mathbb{R}^3$. Show that the trajectories $\mathbf{x}(t)$ of a particle with equations of motion

$$\ddot{\mathbf{x}} = \mathbf{E} + 2\mathbf{B} \wedge \dot{\mathbf{x}}$$

are unparametrised geodesics of some torsion–free connection (which should be given in terms of its connection components) on $\mathbb{R}^3 \times \mathbb{R}$.

3

(a) Explain briefly how a vector field X can be used to define a 1-parameter group of diffeomorphisms ϕ_t.

(b) Let α be a p-form. Prove that the Lie derivative of α can be written

$$\mathcal{L}_X \alpha = i_X d\alpha + d(i_X \alpha)$$

where $i_X \beta$ denotes the operation of contracting the vector field X with the first index of a differential form β. [*You may assume any results derived in lectures.*]

(c) On $\mathbb{R}^3$ with coordinates (x, y, z), define two vector fields X, Y, a 1-form α and a 3-form μ:

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \qquad Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

$$\alpha = y dx - x dy + z dz \qquad \mu = dx \wedge dy \wedge dz$$

(i) Calculate $\mathcal{L}_X \alpha$.

(ii) Let ϕ_t and ψ_t be the 1-parameter families of diffeomorphisms defined by X and Y respectively. Describe the actions of ϕ_t and ψ_t geometrically.

(iii) Calculate $(\phi_t)^* \mu$ and $(\psi_t)^* \alpha$. Hence calculate $\mathcal{L}_X \mu$ and $\mathcal{L}_Y \alpha$. Compare your results with those obtained using the formula in (b) above.

(d) A spacetime admits a Killing vector field V^a and contains a Maxwell field F such that $\mathcal{L}_V F = 0$. Consider the motion of a particle of charge q and rest mass m. Prove that locally there exists a scalar field Φ such that $i_V F = d\Phi$ and hence show that there is a conserved quantity along the particle's worldine. [*The equation of motion of the particle is* $u^b \nabla_b u^a = (q/m) F^a{}_b u^b$.]

4

A spacetime admits an orthonormal basis of 1-forms

$$e^0 = dt \qquad e^1 = A\,dx \qquad e^2 = B\,dy \qquad e^3 = dz$$

where $A = A(t-z)$ and $B = B(t-z)$ are non-zero smooth functions of $t-z$.

(a) Write down the corresponding Lorentzian metric in coordinates (t, x, y, z).

(b) The connection 1-forms are determined uniquely by $de^\mu = -\omega^\mu{}_\nu \wedge e^\nu$. Show that the non-zero connection 1-forms are

$$\omega_{01} = \alpha\, e^1 \qquad \omega_{13} = \beta\, e^1 \qquad \omega_{02} = \gamma\, e^2 \qquad \omega_{23} = \delta\, e^2$$

and those related by $\omega_{\mu\nu} = -\omega_{\nu\mu}$, where $\alpha, \beta, \gamma, \delta$ are functions that you should determine.

(c) Use the equation $\Theta^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$ to determine the curvature 2-forms.

(d) Show that the vacuum Einstein equation $R_{ab} = 0$ reduces to $A''/A + B''/B = 0$.

(e) Explain why such solutions can be interpreted as $+$ polarized gravitational waves. [*Hint: linearize.*]

END OF PAPER