UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 13 June, 2017 1:30 pm to 3:30 pm

PAPER 308

CLASSICAL AND QUANTUM SOLITONS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The ϕ^4 theory in one spatial dimension has Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \lambda (m^2 - \phi^2)^2 \, .$$

By using a Bogomolny rearrangement of the energy for a static field, or otherwise, find the kink solution $\phi_{\rm K}(x)$ and its energy.

A kink moving at velocity $v \ll 1$ can be expressed approximately as $\phi_{\rm K}(x - vt)$. What are the energy and momentum of this moving kink, to lowest non-trivial order in v? Justify your answer. Give an exact formula for a kink moving at constant velocity v.

Consider initial data for ϕ describing two kinks and one antikink at rest, with the kinks located at $x = \pm a$ and the antikink located at $x = 0$, where $\sqrt{2\lambda}\, ma \gg 1$. Use the kink solution $\phi_{\rm K}(x)$ to write down a formula for ϕ representing this initial data. Describe qualitatively the time evolution of the field starting with this initial data.

2

The critically coupled, abelian Higgs theory in $2 + 1$ dimensions has Lagrangian density

$$\mathcal{L} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2}\overline{D_\mu \phi} D^\mu \phi - \frac{1}{8}(1 - \overline{\phi}\phi)^2 \, ,$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ and $D_\mu \phi = \partial_\mu \phi - i a_\mu \phi$. What is meant by a multi-vortex solution in this theory? Derive the field equations of the theory.

A static multi-vortex solution can be obtained by solving the Bogomolny equations

$$D_1 \phi + i D_2 \phi = 0 \, , \qquad (1)$$

$$f_{12} - \frac{1}{2}(1 - \overline{\phi}\phi) = 0 \, . \qquad (2)$$

Show that such a solution also satisfies the static version of the field equations.

Now write the field ϕ in the form $\phi = e^{u + i\chi}$ where u and χ are real fields. Reexpress equation (1) as a relation between the spatial part of the gauge potential and the gradients of u and χ. What can you deduce from this relation, assuming that the gauge is fixed so that the contours of constant χ are everywhere orthogonal to the contours of constant u?

3

Describe how, in the Skyrme model, nucleons are modelled as solitons in a field theory of pions. Discuss the symmetries that occur in the Skyrme model, including chiral symmetry and isospin symmetry.

Describe, with sketches, the Skyrmions with baryon numbers 1, 2, 3 and 4. What are the symmetries of these Skyrmions? Discuss the roles of the symmetry group of the Skyrme model, and the symmetry group of a particular Skyrmion, when that Skyrmion is quantized.

END OF PAPER