UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 9 June, 2017 1:30 pm to 4:30 pm

PAPER 306

STRING THEORY

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The action for the bosonic string in flat spacetime is

$$-\frac{1}{4\pi\alpha'}\int d^2\xi\ \sqrt{-h}\ h^{\mu\nu}\ \partial_\mu X^a\,\partial_\nu X^b\ \eta_{ab}.$$

Derive from first principles the quantization rules for the creation and annihilation operators of the open string assuming that every direction transverse to the string is an ND (Neumann-Dirichlet) direction.

Assuming that only directions transverse to the string are physical (the light cone gauge) and the critical dimension is 26, write down the classical version of the Virasoro operator L_0.

What is the shift in vacuum energy when the string is quantized?

What are the three lowest energy states of such a string?

How do these states fit into representations of the little group?

[*Hint: You may assume that* $\zeta_R(-1) = -\frac{1}{12}$.]

2

The action

$$\frac{-1}{4\pi\alpha'}\int d^2\xi\ \sqrt{-h}\ h^{\mu\nu}\ \partial_\mu X^a\partial_\nu X^b\ g_{ab}(X) + \frac{1}{4\pi}\int d^2\xi\ \sqrt{-h}\ {}^{(2)}R\ \Phi(X)$$

describes the closed bosonic string propagating in a background spacetime with metric $g_{ab}(X)$ and dilaton $\Phi(X)$ and where ${}^{(2)}R$ is the Ricci scalar of the worldsheet metric $h_{\mu\nu}$. Derive the equation that is analogous to the Einstein equation for the gravitational field and the dilaton and which is the condition for the string to be conformally invariant, to lowest order in α'.

Using the contracted Bianchi identities, and the Ricci identities, use your equation to derive an analogue of the Klein-Gordon equation for the dilaton.

[*Hint: You may find it useful to recall that if the worldsheet metric* $h_{\mu\nu}$ *undergoes a conformal transformation*

$$h_{\mu\nu} \to \Omega^2 h_{\mu\nu}$$

then the two-dimensional Ricci scalar ${}^{(2)}R$ *undergoes a corresponding transformation*

$${}^{(2)}R \to \Omega^{-2}({}^{(2)}R - 2\Box\ln\Omega).$$

]

3

The vertex operator for the closed bosonic string is

$$V = g_s \int d^2z \; e^{ip\cdot X(z)},$$

where g_s is the string coupling constant, p is the momentum of the tachyon which is on-shell so that $\alpha' p^2 = 4$ and the tachyon is attached to the string worldsheet at the point z.

Derive the (reduced) four-point tachyon scattering amplitude

$$A^{(4)} = 2\pi g_s^2 \frac{\Gamma(-1-\alpha' s/4)\Gamma(-1-\alpha' t/4)\Gamma(-1-\alpha' u/4)}{\Gamma(2+\alpha' s/4)\Gamma(2+\alpha' t/4)\Gamma(2+\alpha' u/4)},$$

where s, t and u are the Mandelstam variables.

Briefly comment on the analytic structure of this amplitude.

4

The action for the superstring can be taken to be

$$\frac{-1}{4\pi\alpha'} \int d^2\xi \; \sqrt{-h} \; h^{\mu\nu} \; \partial_\mu X^a \partial_\nu X^b \; \eta_{ab} + i\alpha' \; \bar{\psi}^a \gamma^\mu \partial_\mu \psi^b \; \eta_{ab}.$$

Explain what is meant for the spinor ψ^a to be Majorana.

Show that the action is invariant up to a boundary term, under the rigid supersymmetry generated by the constant Majorana spinor ϵ

$$\delta X^a = \sqrt{\frac{\alpha'}{2}} i\bar{\epsilon}\psi^a,$$

$$\delta\psi^a = \sqrt{\frac{1}{2\alpha'}} \gamma^\mu \partial_\mu X^a \; \epsilon.$$

If one is to build a model of the real world, briefly explain why the superstring is a more promising candidate than the bosonic string.

END OF PAPER