UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 6 June, 2017 9:00 am to 12:00 pm

PAPER 305

THE STANDARD MODEL

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) The 4×4 matrix B is defined so that

$$B^{-1}\gamma^{\mu *}B = \begin{cases} \gamma^{\mu} & \mu = 0 \\ -\gamma^{\mu} & \mu = 1,2,3\,. \end{cases}$$

Show that $B^{-1}\gamma^{5*}B = \gamma^5$ where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

(b) The time-reversal operator $\hat{T}$ is anti-linear, $\hat{T}\big(\alpha\,|\phi\rangle + \beta\,|\psi\rangle\big) = \alpha^*\hat{T}\,|\phi\rangle + \beta^*\hat{T}\,|\psi\rangle$. By making an appropriate assumption (which you should state) about how momentum eigenstates transform under $\hat{T}$ and expanding in terms of such eigenstates, show that $\langle \hat{T}\phi|\hat{T}\psi\rangle = \langle\phi|\psi\rangle^*$ for arbitrary scalar states $|\phi\rangle$ and $|\psi\rangle$, i.e. $\hat{T}$ is an anti-unitary operator.

(c) Recall the expansion for a Dirac field $\psi(x)$,

$$\psi(x) = \sum_{p,s}\Big[b^s(p)u^s(p)e^{-ip\cdot x} + d^{s\dagger}(p)v^s(p)e^{ip\cdot x}\Big]\,.$$

Show that under time reversal,

$$\hat{T}\psi(x)\hat{T}^{-1} = B\psi(x_T)\,,$$

where $x_T^{\mu} = (-x^0, \vec{x})$. [The intrinsic phase $\eta_T = 1$ in this question and you may assume that

$$\begin{aligned} \hat{T}b^s(p)\hat{T}^{-1} &= (-1)^{\frac{1}{2}-s}b^{-s}(p_T)\,, \\ \hat{T}d^{s\dagger}(p)\hat{T}^{-1} &= (-1)^{\frac{1}{2}-s}d^{-s\dagger}(p_T)\,, \\ (-1)^{\frac{1}{2}-s}u^{-s*}(p_T) &= \gamma^5 C u^s(p)\,, \\ (-1)^{\frac{1}{2}-s}v^{-s*}(p_T) &= \gamma^5 C v^s(p)\,, \end{aligned}$$

where $p_T^{\mu} = (p^0, -\vec{p})$ and as part of your derivation you should find a relation between B and C.]

(d) For the remainder of this question you may assume that $\hat{T}\bar{\psi}(x)\hat{T}^{-1} = \bar{\psi}(x_T)B^{-1}$.

Consider a $U(1)$ gauge theory with gauge field $A_{\mu}(x)$. Given that the term in the Lagrangian, $gA_{\mu}\bar{\psi}\gamma^{\mu}\psi$, where g is a real constant, leads to an interaction which is time-reversal invariant, determine how $A_{\mu}(x)$ transforms under $\hat{T}$.

How does $ia\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi F_{\mu\nu}$ transform under $\hat{T}$, where a is a real constant, $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right]$ and $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$? Could such an interaction arise in the Standard Model?

(e) Explain briefly why C violation and CP violation are required for baryogenesis. What are the other two conditions required?

2

(a) Consider the electroweak part of the Standard Model with gauge group $SU(2)_L \times U(1)_Y$. The covariant derivative acting on the complex scalar doublet ϕ is

$$D_\mu \phi = \left(\partial_\mu + igW_\mu^a \tau^a + \tfrac{i}{2} g' B_\mu\right)\phi \, ,$$

where W_μ^a are the three $SU(2)_L$ gauge bosons and B_μ is the $U(1)_Y$ gauge boson. Write down the part of the electroweak Lagrangian involving only gauge and scalar fields.

Describe how the gauge symmetry is spontaneously broken to $U(1)_{EM}$ via the Higgs mechanism. In particular, you should relate the gauge fields after symmetry breaking ($W_\mu^\pm$, Z_μ, A_μ) to the original gauge fields, define the Weinberg angle θ_W, show that one gauge boson remains massless and find tree-level expressions for the masses of the other three gauge bosons and the Higgs boson. [*Hint: After giving a brief justification you may consider the expansion*

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} ,$$

where v is a real constant and $H(x)$ is a real scalar field.]

Draw Feynman diagrams for all the tree-level interactions involving both Z and H fields.

(b) The interaction term in the Lagrangian which governs the decay of the Higgs boson, H, to two Z bosons is $\mathcal{L}_{HZZ} = \frac{m_Z^2}{v} Z_\mu Z^\mu H$, where v is the Higgs vacuum expectation value. Without neglecting any masses, derive an expression for the decay rate $\Gamma(H \to ZZ)$ supposing that $m_H > 2m_Z$. [*Hint: The following expressions may be used without proof:*

$$\langle 0 \,| Z_\mu |\, Z(k,s)\rangle = \epsilon_\mu(k,s) \, , \qquad \sum_{Z \text{ spins } s} \epsilon_\mu(k,s)\epsilon_\nu^*(k,s) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{M_Z^2} \, ,$$

and the decay rate for $A(p) \to B(k_1) + C(k_2)$ is,

$$\Gamma(A \to BC) = \frac{1}{2m_A} \int \frac{d^3 k_1}{(2\pi)^3 2k_1^0} \int \frac{d^3 k_2}{(2\pi)^3 2k_2^0} (2\pi)^4 \delta^{(4)}(p - k_1 - k_2) \sum_{spins} |\mathcal{M}|^2 \, ,$$

where m_A is the mass of particle A.]

3

(a) State whether or not each of the following processes is allowed at tree level in the Standard Model:

$$\text{(i) } u\,\bar{d} \to c\,\bar{s} \qquad \text{(ii) } u\,\bar{c} \to d\,\bar{s} \qquad \text{(iii) } b \to u\,e^+\,\nu_e \qquad \text{(iv) } c \to d\,\mu^+\,\nu_\mu .$$

For those which are allowed draw all possible tree-level Feynman diagrams.

(b) Suppose that the chiral structure of the weak charged-current interaction has not been determined and so the part of the effective Lagrangian relevant for $u\,\bar{d} \to c\,\bar{s}$ is

$$\mathcal{L}_{\text{eff}} = -\frac{G_F}{\sqrt{2}} \left[\bar{d}\gamma^\alpha (P + Q\gamma^5) u\right] \left[\bar{c}\gamma_\alpha (P + Q\gamma^5) s\right],$$

where P and Q are real constants and $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}$. Neglecting quark masses, CKM matrix factors and the effects of the strong interaction (e.g. hadronization), show that

$$\frac{d\sigma}{d\Omega}\big(u(p_1)\,\bar{d}(p_2) \to c(k_1)\,\bar{s}(k_2)\big) = F(s)\big[H_1\,(1+\cos^2\theta) + H_2\,\cos\theta\big],$$

where θ is the angle between $\vec{p}_1$ and $\vec{k}_1$, $F(s)$ is some function of $s = (p_1+p_2)^2$ which you should find, and the constants H_1 and H_2 should be expressed in terms of P and Q. [*Hints: Work in the centre of momentum frame. The following expressions may be used without proof:*

$$\begin{aligned}
\mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_n}) &= 0 \quad \textit{for } n \textit{ odd}, \\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}), \\
\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= -4i\epsilon^{\mu\nu\rho\sigma}, \\
\epsilon^{\alpha\beta\sigma\rho}\epsilon_{\alpha\beta\lambda\tau} &= -2(\delta^\sigma_{\ \lambda}\,\delta^\rho_{\ \tau} - \delta^\sigma_{\ \tau}\,\delta^\rho_{\ \lambda}),
\end{aligned}$$

and the differential cross section for $A(p_A) + B(p_B) \to C(p_C) + D(p_D)$ *is,*

$$d\sigma = \frac{1}{|\vec{v}_A - \vec{v}_B|}\frac{1}{4p_A^0 p_B^0}\left(\frac{d^3p_C}{(2\pi)^3 2p_C^0}\right)\left(\frac{d^3p_D}{(2\pi)^3 2p_D^0}\right)(2\pi)^4\delta^{(4)}(p_A+p_B-p_C-p_D)\,\frac{1}{4}\sum_{spins}|\mathcal{M}|^2,$$

*where A, B, C and D are spin-*1/2 *fermions.*]

(c) What happens to your expression at high energy (large s)? Comment on this.

(d) Find an expression for the asymmetry

$$A(\theta) = \frac{\frac{d\sigma}{d\Omega}(\theta) - \frac{d\sigma}{d\Omega}(\theta+\pi)}{\frac{d\sigma}{d\Omega}(\theta) + \frac{d\sigma}{d\Omega}(\theta+\pi)}.$$

Why might the experimental measurement of this asymmetry be a useful way to constrain the chiral structure of the weak charged-current interaction?

4

To leading order, the scale dependence of a renormalized coupling $g(\mu)$ is given by

$$\mu \frac{dg}{d\mu} = -\frac{\beta_0}{16\pi^2} g^3 ,$$

where μ is the renormalization scale and β_0 is a real constant.

(a) Briefly explain the consequences of β_0 being positive (as in QCD) or negative.

(b) For $\alpha_s = g^2/4\pi$ and assuming that $\beta_0 > 0$, derive a expression for $\alpha_s(\mu)$ in terms of an energy scale Λ where α_s diverges.

(c) In QCD, suppose that $\Lambda = \Lambda_4$ for $\mu < m_b$ (where $n_f = 4$) and $\Lambda = \Lambda_5$ for $m_b < \mu < m_t$ (where $n_f = 5$). Here m_b and m_t are respectively the bottom and top quark masses. By requiring that $\alpha_s(\mu)$ is continuous at $\mu = m_b$, show that,

$$\Lambda_5 = \Lambda_4 \left(\frac{m_b}{\Lambda_4} \right)^{-p/r} ,$$

where p and r are positive integers which you should find. [You may assume that $\beta_0 = \frac{11}{3} N - \frac{2}{3} n_f$ in an $SU(N)$ gauge theory with n_f flavours of quarks.]

(d) The cross section for $e^+(p_1)\, e^-(p_2) \to$ hadrons beyond leading order may be written as

$$\sigma = \frac{4\pi\alpha^2}{3q^2}\, 3 \sum_f Q_f^2\, K\left(\alpha_s(\mu^2), q^2/\mu^2\right) ,$$

where Q_f is the charge of quark flavour f, $q = p_1 + p_2$ and at $\mathcal{O}(\alpha_s^2)$

$$K\left(\alpha_s(\mu^2), q^2/\mu^2\right) = 1 + \frac{\alpha_s(\mu^2)}{\pi} + \frac{\alpha_s^2(\mu^2)}{\pi^2} \left[1.99 - 0.11\, n_f - \frac{\beta_0}{4} \ln\left(q^2/\mu^2\right) \right] .$$

One way to choose the arbitrary scale μ is to require that

$$\frac{dK}{d \ln \mu^2} = 0 .$$

By using this prescription with the leading order scale dependence of $\alpha_s(\mu)$, find an expression for μ^2 in terms of n_f and q.

END OF PAPER