UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2017 9:00 am to 12:00 pm

PAPER 304

ADVANCED QUANTUM FIELD THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Let H be the Hamiltonian of a quantum simple harmonic oscillator of unit mass in one dimension, with frequency ω. Let $\mathcal{H}$ be the corresponding Hilbert space. Compute the partition function $\mathcal{Z}(\omega, \beta) = \mathrm{tr}_{\mathcal{H}}(\mathrm{e}^{-\beta H})$ in units with $\hbar = 1$.

(ii) Write this partition function in terms of a formal Euclidean worldline path integral over the space of maps $x : S^1 \to \mathbb{R}$, making clear the role of β and the choice of action $S[x]$.

(iii) Now consider a regularization which is a finite dimensional approximation to this formal path integral, obtained by imposing a hard cut-off on the eigenvalues of the quadratic operator in the wordline action. What is the regularized path integral measure? Show that the regularized partition function is given by

$$\mathcal{Z}_N(\omega, \beta) = \frac{A_N}{\omega} \prod_{n=1}^{N} \left[\omega^2 + \left(\frac{2\pi n}{\beta} \right)^2 \right]^{-1} .$$

where N labels the cut-off and A_N is a constant that is independent of ω. [You need not determine the value of A_N.]

(iv) Consider the ratio

$$\frac{\mathcal{Z}_N(\omega_1, \beta)}{\mathcal{Z}_N(\omega_2, \beta)}$$

of regularized partition functions for harmonic oscillators of different frequencies. By examining the zeros and poles of this ratio as a function of (ω_1, ω_2), show that the limit

$$\lim_{N \to \infty} \mathcal{Z}_N(\omega, \beta)$$

agrees with your answer to part (i), up to an overall constant.

2

A certain four–dimensional Euclidean gauge theory has gauge group G.

(i) What are the quantum numbers (*i.e.* spin, statistics, G-representation) of the ghosts and the Nakanishi–Lautrup field h?

(ii) Now let $G = U(1)$ and let the gauge field A_μ be Hermitian. Construct the ghost and gauge–fixing terms in the action that impose the gauge $\partial^\mu A_\mu + iA^\mu A_\mu = 0$. Why is it necessary to introduce ghosts when working in this gauge?

(iii) What is the contribution to the effective action from integrating out the ghosts?

(iv) Find a theory of matter, whose mass, charge, spin and statistics you should state, such that when A_μ is in the above gauge, the effective action $S_{\text{eff}}[A]$ obtained by integrating out the ghosts and matter fields coincides with the classical Maxwell action. Give an interpretation of this fact.

3

Consider the action

$$S[\phi] = \int d^6x \left[\frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2}\phi^2 + \frac{g_n}{n!}\phi^n \right]$$

for a massive scalar field in six Euclidean dimensions. Give the values of n for which the coupling g_n is *relevant*, *marginal* and *irrelevant* at the classical level.

For the rest of this question, we take $n = 3$ and regularize the theory with a hard momentum cut-off at scale Λ_0.

(i) Draw all 1PI 1-loop Feynman diagrams that contribute to a ϕ^4 term in the effective action at scale Λ, where $\Lambda < \Lambda_0$. Use the Feynman rules to write down the corresponding loop integral for (any) one such diagram.

(ii) To which other terms in the effective action does your loop integral contribute?

(iii) Calculate the contribution of all 1PI 1-loop diagrams to the pure ϕ^4 vertex in the scale Λ effective action. [*The hypersurface area of a unit sphere in six dimensions is* π^3.]

(iv) Up to 1-loop accuracy, which other diagrams contribute to the pure ϕ^4 vertex? Write down the corresponding contributions and check they have correct mass dimensions expected for a contribution to the effective quartic coupling. (You are not required to evaluate any integrals.)

(v) Discuss the behaviour of each of the above loop diagrams as $\Lambda_0 \to \infty$. Discuss the relevance of these considerations to the existence of a continuum ϕ^3 theory in six dimensions (at least perturbatively).

4

Let ψ^a be a Dirac spinor transforming in the adjoint representation of a non-Abelian gauge group G, where $a = 1, \dots, \dim(G)$.

(i) Give an expression for the covariant derivative $\not{D}\psi$ in this representation, in conventions where the generators of the Lie algebra are Hermitian and the gauge field is canonically normalised.

(ii) Assuming the Dirac spinor has mass m and is minimally coupled, write the terms in the $d = 4$ Euclidean action involving ψ. What are the momentum space Feynman rules for its propagator and vertices?

(iii) This Dirac spinor gives a contribution

$$(\Pi^{\text{spinor}}_{\mu\nu})^{ab}(k) = -\frac{8g^2(\mu)\delta^{ab}}{(4\pi)^{d/2}}\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right) C_2(G)\Gamma(2-d/2)\int_0^1 dx\, x(1-x)\left(\frac{\mu^2}{\Delta}\right)^{2-d/2}$$

to the momentum space self-energy of the gluon, to 1-loop accuracy in dimensional regularization. Calculate the contribution of this field to the $\overline{\text{MS}}$ β-function for the Yang–Mills coupling. Hence show that the $d = 4$ Yang–Mills coupling cannot be made marginal to 1-loop accuracy by coupling the pure gauge theory to any number of adjoint valued Dirac spinors.

[*$C_2(G)$ is the quadratic Casimir of the adjoint representation of G, μ is an arbitrary mass scale, $g(\mu)$ is the dimensionless coupling at this scale, and $\Delta := m^2+k^2x(1-x)$. You may assume that*

$$\Gamma(\epsilon/2) \sim \frac{2}{\epsilon} - \gamma_{\text{E}} + \mathcal{O}(\epsilon)$$

as $\epsilon \to 0^+$, where γ_{E} is the Euler–Mascheroni constant.]

END OF PAPER