UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 2 June, 2017 1:30 pm to 3:30 pm

PAPER 303

STATISTICAL FIELD THEORY

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider the Landau free energy density for a magnetic system given by

$$\mathcal{A}(h,\overline{m}) = -h\overline{m} + \frac{1}{2}\mathcal{A}_2\overline{m}^2 + \frac{1}{4}\mathcal{A}_4\overline{m}^4 + \frac{1}{6}\mathcal{A}_6\overline{m}^6\,,$$

where h is the applied external magnetic field. Assume that $\mathcal{A}_6 > 0$ and that $\mathcal{A}_2$ and $\mathcal{A}_4$ depend on the system's temperature T and another external quantity g.

(a) Given values for h, $\mathcal{A}_2$, $\mathcal{A}_4$, and $\mathcal{A}_6$, very briefly explain how can $\mathcal{A}$ be used to determine the equilibrium expectation value of the system's magnetization?

(b) Discuss the behaviour of $\mathcal{A}_2$ and $\mathcal{A}_4$ as the parameters T and g are varied across a second order phase transition, a first order phase transition, and across a tricritical point. Be sure to write down any significant relations between the coefficients of $\mathcal{A}$ at these points.

(c) Determine the critical exponents α, β, γ, and δ as a tricritical point is approached. These exponents can be defined through

$$C \sim |T - T_c|^{-\alpha}\,, \quad m \sim (T_c - T)^{\beta}\,, \quad \chi \sim |T - T_c|^{-\gamma}\,, \quad \text{and} \quad m \sim h^{1/\delta}\,,$$

where C is the specific heat, m the magnetization, and χ the magnetic susceptibility. *[Hints: In the case of the β exponent, we assume T approaches T_c from below. In this question it is sufficient to determine γ from only one side of the $T \to T_c$ limit.]*

(d) Within the context of Landau theory argue that the Helmholtz free energy density $\mathcal{F}$ may be expressed as

$$\mathcal{F} = \frac{|\mathcal{A}_2|^u}{\mathcal{A}_6^v}\,\Phi\left(\frac{\mathcal{A}_4}{|\mathcal{A}_2|^w\mathcal{A}_6^x}, \frac{h\mathcal{A}_6^y}{|\mathcal{A}_2|^z}\right)\,,$$

where Φ is a dimensionless function of 2 variables, with $\Phi(0,0)$ finite and nonzero. You should determine numerical values for the exponents u, v, w, x, y, and z.

(e) At a second order phase transition, what value for the critical exponent α would you predict using the scaling form in Part (d)?

(f) Again using the results of Part (d), what value for α is predicted near a tricritical point?

2

(a) Consider the N-site, one-dimensional spin model, with periodic boundary conditions, described by the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - L \sum_{\langle ij \rangle} \sigma_i^2 \sigma_j^2 - D \sum_i \sigma_i^2 - NK$$

where $i, j \in \{1, 2, 3, \ldots, N\}$, each spin degree-of-freedom $\sigma_i \in \{-1, 0, 1\}$, and $\sum_{\langle ij \rangle}$ denotes a sum over nearest neighbour pairs.

Write down the transfer matrix W for the system. Defining

$$\begin{aligned} x &= e^{-\left(J+L+\frac{D}{2}\right)} \\ y &= e^{-2J} \\ z &= e^{-(J+L+D)} \end{aligned}$$

you should express W in terms of x, y, z, and K alone.

Denote and order the eigenvalues of the transfer matrix as follows $\lambda_1 > \lambda_2 > \lambda_3$. [Do not find explicit expressions for them.] Show how, in the large N limit, the eigenvalues can be used to determine the system's free energy and the magnetization $\langle \sigma_i \rangle$.

Carry out a renormalization group (RG) transformation, "decimating" the spins on alternate sites, and determine the couplings x', y', and z' of the theory on the coarse lattice. [You do not need to explicitly write out an expression for K'.]

(b) In this Part we consider a *different* theory to the one above. Let us consider an unspecified two-dimensional spin system. The only pertinent information you need is that there are couplings u and v, related to the couplings in the Hamiltonian similarly to how x, y, and z are defined in Part (a). Under a real space RG transformation, the couplings transform as $(u, v) \to (u', v')$ with

$$u' = \frac{4u^2}{(1+u^2)^2}, \qquad v' = \frac{v^4}{(1+u^2)^2}.$$

Find the fixed points in the domain $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \infty$. [You may denote by u_* the sole real root of the cubic $u^3 + u^2 + 3u - 1$, and you may use, without proof, the fact that $u_* \approx 0.3$.]

Linearize the RG transformation about a generic fixed point $(\tilde{u}, \tilde{v})$ and use this expression to discuss the nature of any fixed points in the domain $0 < u < 1$, $0 < v < \infty$.

3

Consider the Landau-Ginzburg Hamiltonian describing a real scalar field ϕ in D spacial dimensions

$$H = H_0 + V$$

with

$$\begin{aligned} H_0[\phi, h] &= \int d^D x \left[\frac{1}{2}\,\alpha^{-1}(\nabla\phi)^2 \;+\; \frac{1}{2}\,r_0\,\phi^2 \;-\; h(\mathbf{x})\phi(\mathbf{x})\right] \\ V[\phi] &= \int d^D x\; \frac{1}{4!}\,u_0\,\phi^4\,. \end{aligned}$$

Take as given that $u_0 \geqslant 0$ and $h(\mathbf{x})$ is a slowly varying function of $\mathbf{x}$.

(a) Neglecting the interaction term V, carry out a renormalization group (RG) transformation in Fourier space, eliminating the short wavelength modes with wavevectors $\mathbf{p}$ in the range $\Lambda/b < p \leqslant \Lambda$, where Λ is the maximum wavevector before the transformation, and $b > 1$ is the scale factor associated with the RG transformation. For $h = 0$, show that

$$\alpha^{-1} \to \alpha^{-1} \quad \text{and} \quad r_0 \to b^2 r_0$$

near the Gaussian fixed point. For constant $h \neq 0$ (uniform in space) determine how h behaves under the RG transformation.

(b) Treating the interaction term V as a small perturbation, carry out the same RG transformation as above to find how the couplings (r_0, u_0) behave near the Gaussian fixed point. Discuss the nature of the Gaussian fixed point, paying attention to the dimensionality D.

[Hints: You may denote the following ratio of functional integrals, involving a functional O depending on the long and short wavelength modes, $\tilde{\phi}_<$ and $\tilde{\phi}_>$, as an expection value:

$$\Big\langle O[\tilde{\phi}_<, \tilde{\phi}_>] \Big\rangle_0^{\text{shell}} \equiv \frac{\int \mathcal{D}\tilde{\phi}_>\, O[\tilde{\phi}_<, \tilde{\phi}_>]\, e^{-H_0[\tilde{\phi}_>]}}{\int \mathcal{D}\tilde{\phi}_>\, e^{-H_0[\tilde{\phi}_>]}}\,.$$

It will be convenient to define a propagator for the short wavelength modes

$$\langle \phi_>(\mathbf{x})\, \phi_>(\mathbf{y}) \rangle_0^{\text{shell}} \;=\; G_0^>(r) \;=\; \int_{\Lambda/b}^{\Lambda} \frac{d^D p}{(2\pi)^D}\, \frac{\alpha\, e^{-i\mathbf{p}\cdot\mathbf{x}}}{p^2 + \alpha r_0}\,.$$

You should justify the last equality.]

END OF PAPER