UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 1 June, 2017 9:00 am to 12:00 pm

PAPER 302

SYMMETRIES, FIELDS AND PARTICLES

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Write an essay on the Lie groups $SU(2)$ and $SO(3)$ and the relationship between them. In each case you should include a description of the relevant group manifold and a derivation of the corresponding Lie algebra. You should also discuss the relation between the representation theory of these two groups. *Any general results you need which apply to all matrix Lie groups and Lie algebras may be stated without proof.*

2

Let $\mathfrak{g}$ denote a complex, simple, finite-dimensional Lie algebra. Define a *Cartan sub-algebra* (CSA) $\mathfrak{h}$ of $\mathfrak{g}$ and explain what is meant by a *root* α of $\mathfrak{g}$ and by a *Cartan-Weyl basis* for $\mathfrak{g}$. State without proof the general form of the Lie brackets between the generators of the Cartan-Weyl basis.

Consider the case where $\mathfrak{g} = \mathcal{L}_{\mathbb{C}}(SU(N))$. Consider a basis for the CSA consisting of matrices of the form $H^i = \mathcal{T}^{(i,i)} - \mathcal{T}^{(i+1,i+1)}$ for $i = 1, \ldots, N-1$. Here $\mathcal{T}^{(i,j)}$ denotes the $N \times N$ matrix with componenents

$$\left(\mathcal{T}^{(j,k)}\right)_{\alpha\beta} = \delta_{\alpha j}\delta_{k\beta}$$

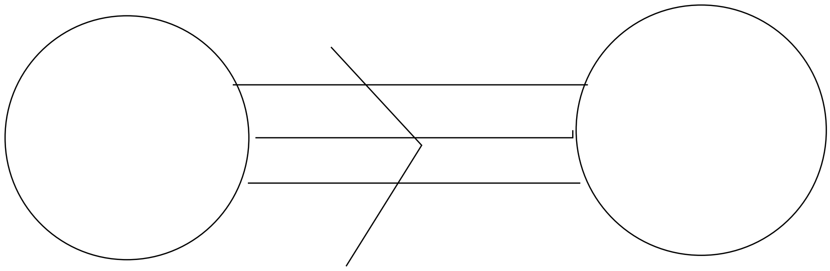
where $j, k, \alpha, \beta = 1, 2, \ldots, N$. Find all the roots of $\mathfrak{g}$ as $(N-1)$-component vectors in this basis and identify the generators of the corresponding Cartan-Weyl basis as $N \times N$ matrices.

Evaluate the Lie brackets of each pair of generators of the Cartan-Weyl basis, giving the result in each case as an explicit linear combination of basis elements.

3

In this question you may use any general results from the theory of complex, simple, finite-dimensional Lie algebras provided they are clearly and correctly stated.

The simple lie algebra G_2 has Dynkin diagram,



What is the angle between the simple roots corresponding to the two nodes of the diagram and what is the ratio of their lengths? In the following, let α denote the longer simple root and β the shorter one.

Given two roots δ and γ of a simple Lie algebra $\mathfrak{g}$, the root string $S_{\gamma,\delta}$ is the set consisting of all roots of the form $\delta + n\gamma$ where n is an integer. Explain briefly what we can deduce about the root string given the inner products of the roots δ and γ. What is the length of the string $S_{\gamma,\delta}$ if both γ and δ are simple roots?

Returning to the case $\mathfrak{g} = G_2$, list the roots in $S_{\alpha,\beta}$ and in $S_{\beta,\alpha}$. By considering further α- and β-strings through these roots, find the full set of roots of G_2. *You may assume that this procedure is sufficent to obtain all the roots but should otherwise explain your reasoning clearly. You should give each root as an integer linear combination of α and β.* Deduce the dimension of the Lie algebra G_2.

Find the fundamental weights ω_1 and ω_2 of G_2. *Here we choose ω_1 to be the fundamental weight having non-zero inner product with the longer simple root α.* Construct the weight set of the irreducible G_2 representation with Dynkin labels $(0, 1)$. Assuming the weights are non-degenerate what is the dimension of the representation?

4

A Lie algebra-valued gauge field,

$$A_\mu : \mathbb{R}^{3,1} \to \mathfrak{g}$$

transforms as $A_\mu \to A_\mu + \delta_X A_\mu$ with,

$$\delta_X A_\mu = -\epsilon \partial_\mu X + \epsilon [X, A_\mu]$$

under an infinitesimal gauge transformation δ_X labeled by a Lie algebra-valued function $X : \mathbb{R}^{3,1} \to \mathfrak{g}$ and a small parameter $\epsilon << 1$. Write down the *field-strength tensor* $F_{\mu\nu}$ corresponding to A_μ and derive its transformation property under the gauge transformation δ_X.

Define the *Killing form* κ on a Lie algebra $\mathfrak{g}$ and prove its invariance property,

$$\kappa\left([Z, X], Y\right) + \kappa\left(X, [Z, Y]\right) = 0 \qquad \forall\ X, Y, Z \in \mathfrak{g}$$

Using the Killing form on $\mathfrak{g}$, write down a *Yang-Mills Lagrangian density* for the above gauge theory and demonstrate its invariance under infinitesimal gauge transformations. Briefly discuss the conditions required for the theory to be physically acceptable and the implied conditions on the Lie algebra $\mathfrak{g}$.

Consider the case $\mathfrak{g} = \mathcal{L}(SU(N))$. Here we define a (non-infinitesimal) transformation of the gauge field according to,

$$A_\mu \to A'_\mu = g A_\mu g^{-1} - \left(\partial_\mu g\right) g^{-1} \qquad (*)$$

where g is a function on spacetime taking values in the Lie group $SU(N)$ and the implied product in the above formula is matrix multiplication. Find the corresponding transformation law for the field-strength tensor under (*). Using the explicit formula,

$$\kappa(X, Y) = \mathrm{Tr}\left(XY\right)$$

for the Killing form on $\mathcal{L}(SU(N))$ show that the corresponding Yang-Mills Lagrangian is invariant under (*).

Explain how the infinitesimal transformation δ_X defined above is related to the finite gauge transformation (*).

END OF PAPER