UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 5 June, 2017 9:00 am to 12:00 pm

PAPER 301

QUANTUM FIELD THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) The Lagrangian density for Maxwell theory is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Show that the theory is gauge invariant. Derive the equation of motion for A_ν. Show that, with the choice of Lorentz gauge $\partial_\mu A^\mu = 0$, the equation of motion reduces to

$$\partial_\mu \partial^\mu A_\nu = 0.$$

(b) Show that the equation of motion in Lorentz gauge follows from

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \left(\partial_\mu A^\mu\right)^2 .$$

Compute the momenta π^μ conjugate to A_μ from this Lagrangian density.

(c) The mode expansions for A_μ and π^μ in the Schrödinger picture are given by

$$\begin{aligned} A_\mu(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda(\vec{p}) \left[a_{\vec{p}}^\lambda e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{\lambda\dagger} e^{-i\vec{p}\cdot\vec{x}} \right] \\ \pi^\mu(\vec{x}) &= i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{|\vec{p}|}{2}} \sum_{\lambda=0}^{3} (\epsilon^\mu)^\lambda(\vec{p}) \left[a_{\vec{p}}^\lambda e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^{\lambda\dagger} e^{-i\vec{p}\cdot\vec{x}} \right] \end{aligned}$$

where ϵ_μ^λ are four 4-vectors satisfying $\epsilon^\lambda \cdot \epsilon^{\lambda'} = \eta^{\lambda\lambda'}$ and $\sum_{\lambda,\lambda'=0}^{3} (\epsilon_\mu)^\lambda (\epsilon_\nu)^{\lambda'} \eta^{\lambda\lambda'} = \eta_{\mu\nu}$, the Minkowski metric. The creation and annihilation operators satisfy the commutation relations

$$\left[a_{\vec{p}}^{\lambda},\ a_{\vec{q}}^{\lambda'\dagger} \right] = -\eta^{\lambda\lambda'} (2\pi)^3 \delta^3(\vec{p} - \vec{q}) \text{ and } \left[a_{\vec{p}}^{\lambda},\ a_{\vec{q}}^{\lambda'} \right] = \left[a_{\vec{p}}^{\lambda\dagger},\ a_{\vec{q}}^{\lambda'\dagger} \right] = 0.$$

Show that this implies canonical commutation relations for A_μ and π^ν.

(d) What is wrong with the Hilbert space spanned by the one particle states $a_{\vec{p}}^{\lambda\dagger}|0\rangle$? Identify the states with longitudinal polarisation and those with transverse polarisation. Write down the Gupta-Bleuler condition. By decomposing the Fock space of some general state $|\Psi\rangle$, find the Gupta-Bleuler condition in terms of the $a_{\vec{p}}^\lambda$. Write a sentence that describes how this solves the problem with one particle states.

2

(a) Write down the Clifford algebra in terms of Dirac matrices γ^μ.

(b) The Dirac representation of γ^μ is (written in 2 by 2 block notation)

$$\tilde{\gamma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \tilde{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where σ^i satisfy $\{\sigma^i,\ \sigma^j\} = 2\delta^{ij}$ and $[\sigma^i,\ \sigma^j] = 2i\epsilon^{ijk}\sigma^k$. Show explicitly that $\tilde{\gamma}^\mu$ satisfy the Clifford algebra.

(c) Find a similarity transform to take the gamma matrices to the chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

(d) A representation $M^{\mu\nu}$ of the Lorentz algebra, satisfies

$$[M^{\mu\nu},\ M^{\rho\sigma}] = M^{\mu\sigma}\eta^{\nu\rho} - M^{\nu\sigma}\eta^{\rho\mu} + M^{\rho\mu}\eta^{\nu\sigma} - M^{\rho\nu}\eta^{\sigma\mu}.$$

Defining $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu,\ \gamma^\nu]$, show the following:
(i) $S^{\mu\nu} = A\gamma^\mu\gamma^\nu + B\eta^{\mu\nu}$,
(ii) $[S^{\mu\nu},\ \gamma^\rho] = C\gamma^\mu\eta^{\nu\rho} + D\gamma^\nu\eta^{\rho\mu}$,
(iii) $[S^{\mu\nu},\ \gamma^\rho\gamma^\sigma] = E\gamma^\mu\eta^{\nu\rho}\gamma^\sigma + F\gamma^\nu\eta^{\mu\rho}\gamma^\sigma + G\gamma^\rho\gamma^\mu\eta^{\nu\sigma} + H\gamma^\rho\gamma^\nu\eta^{\mu\sigma}$,
determining the numbers A, B, C, D, E, F, G, H as you do so.
(iv) Thus verify that $S^{\mu\nu}$ is a representation of the Lorentz algebra.

(e) A Lorentz transformation is defined by $\Lambda = \exp(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma})$, where $M^{\rho\sigma}$ is in the vector representation. How does a Dirac spinor $\psi_\alpha(x)$ transform under this Lorentz transformation?

(f) Consider the rotations S^{jk}, where $j, k \in \{1, 2, 3\}$. Show that

$$S^{jk} \propto \epsilon^{jkl} \begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{pmatrix},$$

finding the constant of proportionality.

(g) Using $\Omega_{ij} = -\epsilon_{ijk}\phi^k$ to describe a spatial rotation, show that $\psi(x) \to O\psi(\Lambda^{-1}x)$, where

$$O = \begin{pmatrix} \exp(\frac{i}{2}\vec{\phi}\cdot\vec{\sigma}) & 0 \\ 0 & \exp(\frac{i}{2}\vec{\phi}\cdot\vec{\sigma}) \end{pmatrix}.$$

Considering a rotation of 2π around the $z-$ axis, $\vec{\phi} = (0,\ 0,\ 2\pi)$, how does $\psi(x)$ transform? How does this compare to a vector representation of the Lorentz algebra?

3

Consider classical field theories where the Lagrangian $\mathcal{L}$ is a function of a real scalar field ϕ. Derive the Euler-Lagrange equation, stating any conditions.

Define a symmetry of a classical field theory of ϕ under the general transformation $\delta\phi = X(\phi)$ in terms of what happens to the Lagrangian density. State Noether's theorem and then prove it for a Lagrangian of the field ϕ. Find a conserved current j^μ.

Derive currents associated to space-time translations. What are these currents traditionally called?

Write down a Lagrangian density for a free, propagating field ϕ of mass m. Use the Euler-Lagrange equation to derive the Klein-Gordon equation. Write down expressions for the conserved quantities associated with space-time translation invariance in terms of $\dot{\phi}$, $\nabla\phi$ and ϕ, giving the interpretation of each conserved quantity.

A complex scalar field ψ has Lagrangian density

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi - V(\psi^*\psi).$$

What internal symmetry does this theory possess? Write down the associated current and conserved charge, and give a potential physical example of what this charge could be.

4

This question is about applications of the time ordering operator T.

(a) Given two operators $O_1(t_1)O_2(t_2)$ evaluated at different times t_1, t_2 define $T\left(O_1(t_1)O_2(t_2)\right)$.

(b) Give the time dependent Schrödinger equation for a state $|\psi\rangle_I$, where subscript I refers to the interaction picture. Using a unitary time evolution operator $U(t, t_0)$, suggest a trial solution for $|\psi(t)\rangle_I$ and find a condition on U from the Schrödinger equation.

Quote Dyson's formula for U, and then show that U satisfies the condition you found above.

(c) A real scalar field has an expansion in terms of annihilation/creation operators $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ip\cdot x} + a^\dagger_{\vec{p}}\, e^{ip\cdot x} \right).$$

What is the value of the commutator $[a_{\vec{p}}, a^\dagger_{\vec{q}}]$? Define the Feynman propagator $\Delta_F(x-y)$ in terms of $\phi(x)$ and $\phi(y)$. Show that

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)},$$

where you should specify how the poles in the integrand are dealt with.

END OF PAPER