UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 12 June, 2017 1:30 pm to 3:30 pm

PAPER 217

GAUSSIAN PROCESSES

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let X be a centred Gaussian random vector in $\mathbb{R}^n, n \in \mathbb{N}$. For $x, y \in \mathbb{R}^n$, denote the Euclidean inner product by $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ and the maximum norm by $\|x\|_\infty = \max_{i \leqslant n} |x_i|$. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable map with gradient $\nabla f(x)$ satisfying $\sup_{x \in \mathbb{R}^n} \|\nabla f(x)\|_\infty < \infty$. Show carefully that for all $\lambda > 0$

$$E\left[e^{\lambda |f(X) - Ef(X)|}\right] \leqslant E\left[e^{\frac{\lambda \pi}{2} |\langle \nabla f(X), Y \rangle|}\right]$$

where Y is an i.i.d. copy of X.

Now suppose there exists a standard normal random variable Z defined on the same probability space as X, Y, such that $|\langle \nabla f(X), Y \rangle| \leqslant |Z|$ holds almost surely. Show that

$$\Pr(|f(X) - Ef(X)| > u) \leqslant 2e^{-\frac{2}{\pi^2}u^2}$$

for all $u > 0$.

2

Let $X = (X(t) : t \in T), T = \mathbb{R}$, be a centred Gaussian process with covariance $\Phi(s,t) = K(s-t), s, t \in T$, where $K : \mathbb{R} \to \mathbb{R}$ is a positive definite, integrable and three-times differentiable function. Show that X has a version $\tilde{X}$ such that the map $t \mapsto \tilde{X}(t)$ is almost surely path-wise differentiable on the interval $(0, 1)$.

[Hint: You may use results from lectures, such as Dudley's theorem on sample-continuity of Gaussian processes, provided they are clearly stated. You may also use the facts that a positive definite function is the Fourier transform of a non-negative function, and that any such function prescribes a proper covariance of a Gaussian process.]

3

Let X be a centred Gaussian random variable X taking values in a separable Banach space B. Define the reproducing kernel Hilbert space (RKHS) H of X. State (without proof) the Cameron-Martin theorem on absolute continuity of laws of shifted Gaussian random variables on B.

Let C be a Borel subset of B that is symmetric about the origin. Show that for every $h \in H$ with RKHS-norm $\|h\|_H$, we have

$$\Pr(X - h \in C) \geqslant e^{-\|h\|_H^2/2} \Pr(X \in C).$$

Now assume that H is dense in B for the $\|\cdot\|_B$-topology and that $\Pr(\|X\|_B \leqslant t) > 0$ for all $t > 0$. Show that the map

$$t \mapsto \Phi(t) = \Pr(\|X\|_B \leqslant t), \quad t \in (0, \infty),$$

is strictly increasing in t.

4

State (without proof) Slepian's comparison lemma for two centred normal random vectors in $\mathbb{R}^n$.

Let $(X_H(t) : t \in [0,1]), (Y_{H'}(t) : t \in [0,1])$ be fractional Brownian motions with Hurst parameters $H > H'$, respectively. Define $T = [0,1] \cap \mathbb{Q}$, where $\mathbb{Q}$ denotes the rational numbers. Show that

$$E \sup_{t \in T} |X_H(t)| \leqslant 4E \sup_{t \in T} |Y_{H'}(t)|.$$

[Recall that a fractional Brownian motion with Hurst parameter $H \in (0,1)$ is a Gaussian processes indexed by $T = [0,1]$ whose covariance is given by

$$\Phi(s,t) = s^{2H} + t^{2H} - |s-t|^{2H}, \; s,t \in T.$$

You may use without proof the fact that sample-continuous versions of fractional Brownian motion exist.]

END OF PAPER