UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 7 June, 2017 1:30 pm to 3:30 pm

PAPER 215

MIXING TIMES OF MARKOV CHAINS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define the *total variation distance* $\|\mu - \nu\|_{\mathrm{tv}}$ for probability distributions μ, ν on a finite set S. Show that if P is the transition matrix of an irreducible, aperiodic Markov chain on a state space S with invariant distribution π, and if $d(t) = \sup_x \|P^t(x,\cdot) - \pi(\cdot)\|_{\mathrm{tv}}$ then $d(t) \leqslant \bar{d}(t)$ where $\bar{d}(t) = \sup_{x,y} \|P^t(x,\cdot) - P^t(y,\cdot)\|_{\mathrm{tv}}$.

(b) Define what is meant by a *coupling* of μ and ν, and show that if (X,Y) is such a coupling then

$$\|\mu - \nu\|_{\mathrm{tv}} \leqslant \mathbb{P}(X \neq Y).$$

(c) Consider two disjoint complete graphs K_n and K'_n on the vertices $\{v_1, \ldots, v_n\}$ and $\{v'_1, \ldots, v'_n\}$ respectively. Let G_n be the graph that results from adding to K_n and K'_n a single vertex w as well as an edge from v_1 to w and another from v'_1 to w. Let X denote simple random walk on G_n, and let τ be the first hitting time of w. Show that if $1 \leqslant i \leqslant n$,

$$\|P^t(v_i,\cdot) - P^t(v'_i,\cdot)\|_{\mathrm{tv}} \leqslant \mathbb{P}_{v_i}(\tau > t).$$

Show that $\mathbb{E}(\tau) \leqslant n(n-1)$, and deduce that $t_{\mathrm{mix}} = O(n^2)$, where t_{mix} denotes the mixing time of X. [*Hint:* for $1 \leqslant i \leqslant n$, call the vertices v_i and v'_i related. If two random walks start from unrelated vertices in K_n and K'_n respectively, you can start by showing briefly that they can be moved to related vertices in the next step with probability $1 - O(1/n)$.]

2

(a) Define the notion of *mixing time* $t_{\text{mix}}(\alpha)$ *at level* $\alpha \in (0,1)$ of an irreducible, aperiodic Markov chain on a finite set S and explain what is meant by the cutoff phenomenon.

(b) Let $(X_t, t = 0,1,\ldots)$ be an irreducible, aperiodic and reversible Markov chain on a finite state space S with invariant distribution $\pi(y), y \in S$. Show that if $P^t(x,y)$ denote the t-step transition probabilities of the chain,

$$\frac{P^t(x,y)}{\pi(y)} = \sum_{j=1}^{n} \lambda_j^t f_j(x) f_j(y)$$

where λ_j are the eigenvalues of the transition matrix P and f_j are functions which you should specify. [You can assume without proof that there exists an orthonormal basis of eigenfunctions for a suitable inner product].

(c) In the setup of (b), give the definition of the *absolute spectral gap* γ_* of the chain, as well as that of the *relaxation time* t_{rel}, and show that

$$(t_{\text{rel}} - 1) \log\left(\frac{1}{2\varepsilon}\right) \leqslant t_{\text{mix}}(\varepsilon).$$

(You may use without proof the inequality $-(1-x)\log(1-x) \leqslant x$, valid for all $x \in [0,1)$.)

(d) Suppose that $\{X^n\}_{n\geqslant 1} = \{(X_n(t), t = 0,1,\ldots)\}_{n\geqslant 1}$ is a family of Markov chains satisfying the cutoff phenomenon. Denote by $\gamma = \gamma_n$, $t_{\text{mix}} = t_{\text{mix}}^n$ the spectral gap and the mixing time at level $(1/4)$ of X^n respectively, and suppose that $t_{\text{mix}} \to \infty$. Then show that $\gamma\, t_{\text{mix}} \to \infty$ as $n \to \infty$. (This is called Peres' product condition.)

Show furthermore by considering the case of the complete graph K_n on n vertices that the condition "$t_{\text{mix}} \to \infty$" above cannot be removed to obtain the same conclusion.

3

(a) Consider an irreducible, aperiodic and reversible Markov chain on a state S, and let π be the equilibrium distribution. Explain what is meant by a *Poincaré inequality* with constant $C > 0$. State a formula (no proof required) expressing the spectral gap γ as the solution of a variational problem. How is this related to Poincaré inequalities?

(b) State and prove a theorem demonstrating the use of the *canonical paths method* to obtain a Poincaré inequality.

(c) Consider the hypercube $H_n = \{-1, +1\}^n$ and let $(X_t, t = 0, 1, \ldots)$ denote a lazy random walk on H_n. Hence at each time step $t = 0, 1, \ldots$, a coordinate $1 \leqslant i \leqslant n$ is chosen uniformly at random, and the ith coordinate of X_t is flipped with probability $1/2$.

Use the canonical paths method to establish a Poincaré inequality with constant $C = 2n^2$. [*Hint*: change bits one at a time]. Hence deduce that the spectral gap γ satisfies $\gamma \geqslant 1/(2n^2)$.

(d) Compute the eigenvalues of this chain exactly. How sharp is the estimate obtained in (c)? [*Hint:* for $J \subset \{1, \ldots, n\}$, set $f_J(x) = \prod_{j \in J} x_j$ if $x = (x_1, \ldots, x_n) \in H_n$.]

4

(a) Let P be the transition matrix of an irreducible, aperiodic and reversible Markov chain on a finite state space S of size n with invariant distribution $(\pi(x))_{x \in S}$, with eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$. Define the *Dirichlet form* $\mathcal{E}(f, f)$ associated to P, and give without proof an equivalent expression. State and prove the *variational characterisation* of the spectral gap in terms of $\mathcal{E}(f, f)$.

(b) Consider the Markov chain of "random adjacent transpositions" on the permutation group S_n of order n. This is the Markov chain defined by $P(x, y) = p(x^{-1}y)$ and $p(s) = 1/n$ if s is the identity, or $s = (i, i+1)$ for $1 \leqslant i \leqslant n-1$, and $p(s) = 0$ otherwise. (Here, the notation (i, j) refers to the transposition of i and j.) By considering the function $f : S_n \to \mathbb{R}$ defined by $f(\sigma) = \sigma^{-1}(i)$, show that as $n \to \infty$,

$$\gamma \leqslant \frac{6}{n^3}(1 + o(1)).$$

[You can use without proof that if U is a uniform random variable on $(0, 1)$ then $\mathrm{Var}\ (U) = 1/6$, and that if $U_n \in [0, 1]$ converges to U in distribution, then $\mathrm{var}(U_n) \to 1/6$].

(c) Stating carefully any theorem you use, show conversely that

$$\gamma \geqslant 1/(2n^3).$$

[You can use without proof that for the "random transpositions" walk on S_n, the corresponding spectral gap $\tilde{\gamma}$ is equal to $\tilde{\gamma} = 2/n$. We recall that random transpositions are defined by setting the kernel $\tilde{p}$ to be: $\tilde{p}(s) = 1/n$ when s is the identity; $\tilde{p}(s) = 1/\binom{n}{2}$ for any transposition s; and $\tilde{p}(s) = 0$ otherwise.]

END OF PAPER