UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 6 June, 2017 1:30 pm to 3:30 pm

PAPER 214

PERCOLATION AND RANDOM WALKS ON GRAPHS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let $(x_n)_{n \geqslant 0}$ be a real sequence satisfying $x_{n+m} \leqslant x_n + x_m$ for all $n, m \geqslant 1$. Prove that the limit of x_n/n as $n \to \infty$ exists and satisfies

$$\lim_{n\to\infty} \frac{x_n}{n} = \inf_k \left\{ \frac{x_k}{k} \right\}.$$

(b) Consider bond percolation on $\mathbb{Z}^2$ with parameter $p \in (0,1)$. For all $k \geqslant 1$ define the tube

$$T_k = \{(x_1, x_2) \in \mathbb{Z}^2 : |x_2| \leqslant k\}.$$

(i) Prove that for all $k \geqslant 1$ the limit

$$f_k(p) = \lim_{n\to\infty} \left(-\frac{1}{n} \log \mathbb{P}_p((0,0) \longleftrightarrow (n,0) \quad \text{in } T_k) \right).$$

exists and satisfies for all n and k

$$\mathbb{P}_p((0,0) \longleftrightarrow (n,0) \quad \text{in } T_k) \leqslant e^{-n f_k(p)}.$$

(ii) Prove that the sequence $(f_k(p))_k$ has a limit as $k \to \infty$.

(iii) Prove that $f_k(p) > 0$ for all $k \geqslant 1$.

2

(a) Define bond percolation on $\mathbb{Z}^d$ and the critical probability $p_c(d)$.

(b) Prove that $p_c(2) = 1/2$ and $\theta(1/2) = 0$.

(You can assume exponential decay in the subcritical regime and the uniqueness of the infinite cluster in the supercritical case.)

3

Let $G = (V, E)$ be a finite connected unweighted graph on n vertices. Let a and z be two distinguished vertices.

(a) Let T be a uniform spanning tree. By considering a suitable unit flow, prove that for all edges $e \in E$,

$$\mathbb{P}(e \in T) = R_{\text{eff}}(e),$$

where $R_{\text{eff}}(e)$ is the effective resistance between the endpoints of e.

(You do not need to prove that your flow satisfies Kirchhoff's laws.)

Deduce Foster's theorem:

$$\sum_{e \in E} R_{\text{eff}}(e) = n - 1.$$

(b) Suppose a random walk X on G starts from a. Let τ be the first time it returns to a after having first visited z. Let $S(x, y)$ be the number of times up to time τ that the walk traverses the edge $e = (x, y)$ in the direction from x to y. Prove that

$$\mathbb{E}[S(x, y)] = R_{\text{eff}}(a, z).$$

(You can use results from the course provided you state them clearly.)

(c) Finally prove that

$$\mathbb{E}\left[\sum_{k=0}^{\tau-1} R_{\text{eff}}(X_k, X_{k+1})\right] = 2(n-1)R_{\text{eff}}(a, z).$$

4

Let G be a finite connected graph.

(a) Define the term uniform spanning tree of G.

(b) Describe Wilson's algorithm for generating a uniform spanning tree of G.

Explain the meaning of the term "uniform spanning tree of an infinite recurrent graph".

(c) We define a loop erased random walk from x to y in G to have the law of a simple random walk started at x until it first hits y deleting the loops as we encounter them.

Show that the set of edges a loop erased random walk from x to y crosses has the same distribution as the set of edges crossed by a loop erased random walk from y to x.

END OF PAPER