UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 9 June, 2017 1:30 pm to 3:30 pm

PAPER 203

SCHRAMM-LOEWNER EVOLUTIONS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Give the definition of a compact $\mathbb{H}$-hull A and its half-plane capacity $\mathrm{hcap}(A)$.

(b) Prove that $\mathrm{hcap}(A) = \lim_{y\to\infty} y\mathbb{E}_{iy}[\mathrm{Im}(B_\tau)]$ where B is a complex Brownian motion and τ is its first exit time from $\mathbb{H} \setminus A$. Deduce that $\mathrm{hcap}(A) \geqslant 0$. [You may assume that $\mathrm{hcap}(A)$ is real-valued.]

(c) Establish the following properties of hcap:

(i) $\mathrm{hcap}(A + x) = \mathrm{hcap}(A)$ for all $x \in \mathbb{R}$,
(ii) $\mathrm{hcap}(rA) = r^2\mathrm{hcap}(A)$ for all $r \geqslant 0$,
(iii) $\mathrm{hcap}(A) \leqslant \mathrm{hcap}(B)$ for compact $\mathbb{H}$-hulls A, B with $A \subseteq B$.

(d) Compute $\mathrm{hcap}([0, i])$, $\mathrm{hcap}(\mathbb{H} \cap \mathbb{D})$, and prove that $\mathrm{hcap}(A) \leqslant \mathrm{diam}(A)^2$ for any compact $\mathbb{H}$-hull A.

2

(a) Explain what it means for a family of compact $\mathbb{H}$-hulls to be:

(i) Non-decreasing,
(ii) Locally growing,
(iii) Parameterized by half-plane capacity.

(b) Prove or disprove: if (A_n) is a sequence of compact $\mathbb{H}$-hulls and $\mathrm{hcap}(A_n) \to 0$ as $n \to \infty$, then $\mathrm{diam}(A_n) \to 0$ as $n \to \infty$. [You may use results from lectures provided you state them clearly.]

(c) Define the Dirichlet inner product and prove that it is conformally invariant.

(d) Define the space $H_0^1(D)$. Show that if $U \subseteq D$ is open and $H_{\mathrm{supp}} = H_0^1(U)$ and H_{harm} denotes those $H_0^1(D)$ functions which are harmonic in U, then H_{supp} and H_{harm} are orthogonal subspaces of $H_0^1(D)$. [You do not need to prove that $H_{\mathrm{supp}} \oplus H_{\mathrm{harm}}$ spans $H_0^1(D)$.]

3

(a) State the Koebe-1/4 theorem.

(b) Suppose that $D, \widetilde{D}$ are domains in $\mathbb{C}$, $z \in D$, $\widetilde{z} \in \widetilde{D}$, and $f \colon D \to \widetilde{D}$ is a conformal transformation with $f(z) = \widetilde{z}$. Show that

$$\frac{\widetilde{d}}{4d} \leqslant |f'(z)| \leqslant \frac{4\widetilde{d}}{d}$$

where $d = \operatorname{dist}(z, \partial D)$ and $\widetilde{d} = \operatorname{dist}(\widetilde{z}, \partial \widetilde{D})$.

(c) Show that SLE_κ is space-filling for $\kappa > 8$. [You may assume without proof that

$$M_t = |g_t'(z)|^{(8-\kappa+\rho)\rho/(4\kappa)} \Upsilon_t^{\rho(\rho+8)/(8\kappa)} S_t^{-\rho/\kappa}$$

is a continuous local martingale for each $\rho > 0$ where $\Upsilon_t = \operatorname{Im}(g_t(z))/|g_t'(z)|$, $S_t = \sin(\arg(g_t(z) - U_t))$, and (g_t) is the Loewner flow driven by $U_t = \sqrt{\kappa} B_t$ for B a standard Brownian motion.]

4

(a) Explain what it means for SLE to satisfy the restriction property.

(b) Show that a simple SLE curve γ in $\mathbb{H}$ from 0 to ∞ satifsfies restriction if there exists $\alpha > 0$ such that

$$\mathbb{P}[\gamma[0,\infty) \cap A = \emptyset] = (g_A'(0))^\alpha$$

for all compact $\mathbb{H}$-hulls A with $0 \notin \overline{A}$ and g_A the unique conformal transformation $\mathbb{H} \setminus A \to \mathbb{H}$ with $g_A(z) - z \to 0$ as $z \to \infty$.

(c) Prove that SLE_κ for $\kappa \leqslant 4$ does not intersect the domain boundary. [You may use results from lectures about Bessel processes provided you state them clearly.]

(d) Show for each $z \in \mathbb{H}$ that $\arg(g_t(z) - U_t)$ is a continuous martingale for SLE_4.

END OF PAPER