UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 6 June, 2017 9:00 am to 12:00 pm

PAPER 202

STOCHASTIC CALCULUS AND APPLICATIONS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Explain what is meant for a continuous martingale to be *bounded in* L^2. Define the *previsible* σ*-algebra* $\mathcal{P}$.

(b) Show that if M is a continuous L^2-bounded martingale, then the norms

$$\|M\| := \|M_\infty\|_{L^2} \qquad \text{and} \qquad \|\!|M|\!\| := \left\| \sup_{t \geqslant 0} |M_t| \right\|_{L^2}$$

are equivalent.

(c) Let H be a simple process and M be a continuous L^2-bounded martingale. Define the *Itô integral* $H \cdot M$.

(d) State *Itô's isometry*, carefully defining all the spaces involved. [You may assume existence of the quadratic variation].

(e) Recall that if M is a continuous local martingale, then $[M]$ denotes the unique continuous, adapted, non-decreasing process such that $[M]_0 = 0$ and $M^2 - [M]$ is a continuous local martingale. Show that if H is a bounded previsible process and M is a continuous bounded martingale, then $[H \cdot M] = H^2 \cdot [M]$.

2

Let M, N be continuous local martingales.

(a) Show that there exists a unique continuous, adapted, finite variation process $[M, N]$ such that $[M, N]_0 = 0$ and $MN - [M, N]$ is a continuous local martingale. [You may use facts about the quadratic variation, provided you state them clearly].

(b) Let $\mathbb{P}$ and $\tilde{\mathbb{P}}$ be two probability measures on the same space. Explain what it means to say that $\tilde{\mathbb{P}}$ is *absolutely continuous* with respect to $\mathbb{P}$.

(c) Recall that if M, N are continuous local martingales, and for $n \geqslant 1$ we let

$$C_t^n := \sum_{k=0}^{\lceil 2^n t \rceil - 1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})(N_{(k+1)2^{-n}} - N_{k2^{-n}}),$$

then $C^n \to [M, N]$ u.c.p. as $n \to \infty$. Use this to show that if M, N are continuous local martingales under $\mathbb{P}$, and $\tilde{\mathbb{P}}$ is absolutely continuous with respect to $\mathbb{P}$, then the covariation process $[M, N]$ is the same under $\tilde{\mathbb{P}}$ and under $\mathbb{P}$. [If you want to use that the same holds for the quadratic variation, you should prove it].

(d) Let X, Y be continuous semimartingales, and consider the Itô and Stratonovich integrals

$$I_t = \int_0^t X_s dY_s, \qquad S_t = \int_0^t X_s \partial Y_s.$$

What is the relationship between I_t and S_t?

(e) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is C^3, then $\partial f(X_t) = f'(X_t) \partial X_t$. [You may assume Itô's formula].

3

Consider the SDE $dX_t = b(X_t)dt + \sigma(X_t)dB_t$.

(a) Define *strong* and *weak solutions* to the above SDE. Define *pathwise uniqueness* and *uniqueness in law.*

(b) Give an example (without proof) of an SDE with uniqueness in law but not pathwise uniqueness.

(c) Let B be a standard 1-dimensional Brownian motion on a given probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. For $\mu > 0$, define

$$Z_t = e^{\mu B_t - \mu^2 t/2}.$$

Show that Z is UI on $[0, T]$ for any fixed $T > 0$. [You may use any result from the course, provided you state it carefully.]

(d) Deduce that for any $T > 0$ there exists a probability measure $\mathbb{P}^T$ on the same space such that $\mathbb{P}^T \ll \mathbb{P}$ and $(B_t - \mu t)_{t \leqslant T}$ is a Brownian motion on $[0, T]$.

(e) Can you find a probability measure $\mathbb{P}^\infty$ on the same space such that $\mathbb{P}^\infty \ll \mathbb{P}$ and $(B_t - \mu t)_{t \geqslant 0}$ is a Brownian motion on $[0, \infty)$? Explain. [You may use standard facts about Brownian motion].

4

Let $B = (B_t)_{t \geqslant 0}$ and $\vartheta = (\vartheta_t)_{t \geqslant 0}$ be 1-dimensional Brownian motions defined on the same probability space, and set

$$M_t = e^{B_t + i\vartheta_t} \qquad \text{for } t \geqslant 0,$$

where $i = \sqrt{-1}$. Let X and Y denote the real and imaginary part of M respectively.

(a) Obtain SDEs for X and Y, and explain why they are continuous local martingales.

(b) Show that $[X]_t = [Y]_t$ and $[X, Y]_t = 0$ for all $t \geqslant 0$.

(c) Show that $[X]_\infty = +\infty$. [You may assume recurrence of 1-dimensional Brownian motion].

(d) Show that X is a time-changed Brownian motion starting from 1, and determine the time change. State carefully any result you use.

5

(a) Define a *locally defined process*. Define a *local solution* to the SDE $dX_t = b(X_t)dt + \sigma(X_t)dB_t$. Explain what is meant for a local solution to be *maximal*.

(b) Consider the SDE

$$\begin{cases} dX_t = \sqrt{X_t}dB_t \\ X_0 = x_0 > 0, \end{cases} \tag{1}$$

on $U = (0, +\infty)$. Let (X, T) denote its unique maximal solution, which you may assume satisfies $T = \lim_{n\to\infty} T_{1/n}$, where $T_{1/n} := \inf\{t \geqslant 0 : X_t \leqslant 1/n\}$.

Derive an SDE for $Y_t = \sqrt{X_t}$.

(c) Let $W_t = \sqrt{x_0} + \frac{B_t}{2}$. Show that $\mathbb{P}(W_t \geqslant Y_t \text{ for all } t \in [0, T)) = 1$.

(d) Deduce that $\mathbb{P}(T < \infty) = 1$. [You may assume recurrence of one-dimensional Brownian motion].

(e) Suppose now we consider, for $\alpha \in (0, 1]$, the SDE

$$\begin{cases} dX_t = X_t^{\alpha} dB_t \\ X_0 = x_0 > 0, \end{cases}$$

on $U = (0, \infty)$. Let again (X, T) denote its unique maximal solution, which you may assume satisfies $T = \lim_{n\to\infty} \inf\{t \geqslant 0 : X_t \leqslant 1/n\}$. What can you say about $\mathbb{P}(T < \infty)$? *(Hint: Try to generalise the above strategy for $\alpha \in (0, 1)$. Treat the case $\alpha = 1$ separately.)*

6 Let $f \in C_b^2(\mathbb{R})$, and let $u \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ solve

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + \dfrac{u(t,x)}{1+x^2}, \\ u(0,x) = f(x). \end{cases}$$

(a) Let B be a standard Brownian motion on $\mathbb{R}$ starting from $x \in \mathbb{R}$. Show that, for any fixed $T > 0$, the process

$$M_t = u(T-t, B_t) \exp\Big(\int_0^t \frac{1}{1+B_s^2} ds\Big)$$

is a true martingale on $[0, T]$.

(b) Deduce that

$$u(t,x) = \mathbb{E}_x\Big(f(B_t) \exp\Big(\int_0^t \frac{1}{1+B_s^2} ds\Big)\Big).$$

(c) Assume further that there exists a constant $C > 0$ such that $f(x) \geqslant C > 0$ for all $x \in \mathbb{R}$. Use the above formula to show that $u(x,t) \to +\infty$ as $t \to \infty$, for all $x \in \mathbb{R}$.

END OF PAPER