UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 2 June, 2017 1:30 pm to 4:30 pm

PAPER 201

ADVANCED PROBABILITY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) What does it mean to say that a random process $(M_n)_{n\geqslant 0}$ is a martingale?

(b) State Doob's L^2 inequality.

(c) Let $(X_n : n \in \mathbb{N})$ be a sequence of independent, identically distributed random variables of finite mean m and finite variance σ^2. Set $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \in \mathbb{N}$. Show that, for all $n \in \mathbb{N}$,

$$\mathbb{E}\left(\sup_{k\leqslant n} |S_k - mk|^2\right) \leqslant 4\sigma^2 n.$$

(d) Consider the linear interpolation $(S_t)_{t\geqslant 0}$ given by

$$S_{n+t} = (1-t)S_n + tS_{n+1}, \quad n \in \mathbb{Z}^+, \quad t \in [0,1].$$

Set

$$S_t^{(N)} = N^{-1} S_{Nt}$$

and write μ_N for the distribution of $(S_t^{(N)})_{t\geqslant 0}$ on $C([0,\infty),\mathbb{R})$. Show that the sequence $(\mu_N : N \in \mathbb{N})$ converges weakly on $C([0,\infty),\mathbb{R})$ and determine its limit.

2

(a) Let $(M_n)_{n\geqslant 0}$ be an integrable discrete-time random process, adapted to a filtration $(\mathcal{F}_n)_{n\geqslant 0}$. Show that the following conditions are equivalent:

(i) $(M_n)_{n\geqslant 0}$ is a martingale,

(ii) $\mathbb{E}(M_T) = \mathbb{E}(M_0)$ for all bounded stopping times T.

(b) Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables, each taking the value -1 with probability $6/7$ and taking the value 2 otherwise. Set $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geqslant 1$. Fix $m \in \mathbb{N}$ and set

$$T = T_m = \inf\{n \geqslant 1 : |S_n| \geqslant m\}.$$

Show that $\mathbb{E}(2^{S_T}) = 1$.

(c) Find an equation relating $\mathbb{E}(T)$ and $\mathbb{E}(S_T)$.

(d) Deduce from the equations found in (b) and (c) that $\mathbb{E}(T_m)/m \to 7/4$ as $m \to \infty$.

3

(a) State the L^1 martingale convergence theorem.

(b) State the L^p martingale convergence theorem, for $p \in (1, \infty)$.

(c) Let f be a function $[0,1]$. Suppose that f satisfies the Lipschitz condition

$$|f(x) - f(y)| \leqslant K|x-y| \quad \text{for all} \quad x, y \in [0,1]$$

for some constant $K < \infty$. Show that there exists a bounded measurable function g on $[0,1]$ such that

$$f(x) = f(0) + \int_0^x g(t)dt \quad \text{for all} \quad x \in [0,1].$$

(d) Given a continuous function f on $[0,1]$, define, for $n \geqslant 0$ and $\lambda < \infty$,

$$I_n(\lambda) = \{k \in \{0, 1, \ldots, 2^n - 1\} : |f((k+1)2^{-n}) - f(k2^{-n})| \geqslant \lambda 2^{-n}\}$$

and set

$$V_n(f, \lambda) = \sum_{k \in I_n(\lambda)} |f((k+1)2^{-n}) - f(k2^{-n})|.$$

Suppose that f satisfies

$$\sup_{n \geqslant 0} V_n(f, \lambda) \to 0 \quad \text{as} \quad \lambda \to \infty.$$

Show that there exists a function g on $[0,1]$, integrable with respect to Lebesgue measure, such that

$$f(x) = f(0) + \int_0^x g(t)dt \quad \text{for all} \quad x \in [0,1].$$

4

(a) Let $(\xi_t)_{t \in D}$ be a random process, indexed by the set

$$D = \{k2^{-n} : n \in \mathbb{N}, k = 0, 1, \ldots, 2^n\}.$$

Suppose that, for some $p \in (1, \infty)$ and some $\beta \in (1/p, 1]$, there is a constant $C < \infty$ such that

$$\|\xi_s - \xi_t\|_p \leqslant C|s - t|^\beta, \quad s, t \in D$$

where $\|.\|_p$ denotes the usual $L^p(\mathbb{P})$-norm. Show that, for some range of $\alpha \in (0, 1]$, to be specified, the following series converges in $L^p(\mathbb{P})$

$$K_\alpha = 2 \sum_{n=0}^{\infty} 2^{n\alpha} \sup_{k \in \{0,1,\ldots,2^n-1\}} |\xi_{(k+1)2^{-n}} - \xi_{k2^{-n}}|.$$

(b) Hence show that there is a continuous random process $(X_t)_{t \in [0,1]}$ such that $X_t = \xi_t$ almost surely, for all $t \in D$.

(c) Consider now the case where $(\xi_t)_{t \in D}$ is a zero-mean Gaussian process with $\mathbb{E}(\xi_s \xi_t) = \min\{s, t\}$ for all s, t. Show that the process $(X_t)_{t \in [0,1]}$ in (b) may be chosen to have paths which are Hölder continuous of every exponent $\alpha < 1/2$.

5

Let $B = (B_t)_{t \geqslant 0}$ be a three-dimensional Brownian motion starting from x. Let $r, R \in (0, \infty)$ with $r < R$.

(a) Suppose that $|x| \in (r, R)$. Show that

$$\mathbb{P}_x(|B| \text{ hits } r \text{ before } R) = (|x|^{-1} - R^{-1})/(r^{-1} - R^{-1}).$$

(b) Show that, for $|x| \in (r, \infty)$,

$$\mathbb{P}_x(|B_t| = r \text{ for some } t \geqslant 0) < 1.$$

(c) Show on the other hand that, for $r > 0$ and all x,

$$\mathbb{P}_x(|B_t + z| = r \text{ for some } t \geqslant 0 \text{ and some } z \in \mathbb{Z}^3) = 1.$$

6

Let $(B_t)_{t \geqslant 0}$ be a Brownian motion in $\mathbb{R}$, starting from 0. For $a \geqslant 0$, set

$$H_a = \inf\{t \geqslant 0 : B_t \geqslant a\}.$$

(a) Show that $H_a < \infty$ almost surely and, for all $u \geqslant 0$,

$$\mathbb{E}(e^{ua - u^2 H_a/2}) = 1.$$

(b) Show that $T_a = H_a$ almost surely, for all $a \geqslant 0$, where

$$T_a = \inf\{t \geqslant 0 : B_t > a\}.$$

(c) Is it true that $\mathbb{P}(T_a = H_a \text{ for all } a \geqslant 0) = 1$? Justify your answer.

(d) Let $(W_t)_{t \geqslant 0}$ be a Brownian motion, starting from 0, independent of $(B_t)_{t \geqslant 0}$. Set

$$X_a = W_{T_a}, \quad a \geqslant 0.$$

Show that $(X_a)_{a \geqslant 0}$ is a Lévy process.

(e) Determine the characteristic exponent of $(X_a)_{a \geqslant 0}$.

[You may use without proof any result from the course, provided you state it clearly.]

END OF PAPER